A Bellman approach for regional optimal control problems in ^{N}

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A Bellman approach for regional optimal control problems in $\mathbb{R}^N$

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Abstract

This article is a continuation of a previous work where we studied infinite horizon control problems for which the dynamic, running cost and control space may be different in two half-spaces of some Euclidian space $\mathbb{R}^N$. In this article we extend our results in several directions: (i) to more general domains; (ii) by considering finite horizon control problems; (iii) by weakening the controlability assumptions. We use a Bellman approach and our main results are to identify the right Hamilton-Jacobi-Bellman Equation (and in particular the right conditions to be put on the interfaces separating the regions where the dynamic and running cost are different) and to provide the maximal and minimal solutions, as well as conditions for uniqueness. We also provide stability results for such equations.

Key-words: Optimal control, discontinuous dynamic, Bellman Equation, viscosity solutions.
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1 Introduction

This article is a continuation of [6] where we studied infinite horizon control problems for which the dynamic, running cost and control space may be different in two half-spaces of some Euclidian space $\mathbb{R}^N$. This study was made through the Bellman approach and our main results where to identify the right Hamilton-Jacobi-Bellman Equation (and in particular the right conditions to be put on the hyperplane separating the regions where the dynamic and running cost are different) and to provide the maximal and minimal solutions, as well as conditions for uniqueness. The aim of the present paper is three-fold: (i) to extend these results to more general domains; (ii) to consider also finite horizon control problems; (iii) last but not least, to weaken the controlability assumption made in [6]. We also emphasize the stability properties for such equations which are a little bit different from the classical ones.

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To be more specific, we recall that, in the classical theory (see for example Lions [30], Fleming & Soner [24], Bardi & Capuzzo Dolcetta [4]), Hamilton-Jacobi-Bellman Equation for finite horizon control problems in the whole space $\mathbb{R}^N$ have the form

$$u_t + H(x,t,Du) = 0 \quad \text{in } \mathbb{R}^N \times (0,T)\,, \tag{1.1}$$

where the Hamiltonian H is typically given by

$$H(x,t,p) := \sup_{\alpha \in A} \big\{ -b(x,t,\alpha)\cdot p - l(x,t,\alpha) \big\}\,. \tag{1.2}$$

The control space A is assumed to be compact, the dynamic b and running cost l are supposed to be continuous functions which are Lipschitz continuous in x, so that H is continuous and has suitable properties ensuring existence and uniqueness of a solution to (1.1).

In this paper, as we already mentioned above, we have different dynamics and running costs in different regions. In other words, the functions b and l are no longer assumed to be continous anymore when crossing the boundaries of the different regions, which implies that the Hamiltonian H in (1.2) also presents discontinuities. Hence, getting suitable comparison and uniqueness results for (1.1) in this setting is not obvious at all and the aim of this paper is to give precise answers to these questions.

To be more precise, we are going to decompose $\mathbb{R}^N$ using a collection $(\Omega_i)_{i\in I}$ of regular open subsets of $\mathbb{R}^N$ such that each point $x \in \mathbb{R}^N$ either lies inside one (and only one) Ω_i, or is located on the boundary of exactly two sets Ω_i. Because of the (regularity) assumptions we are going to use, we can in fact reduce this collection to two domains Ω_1, Ω_2 : we refer to Section 6 for comments on this reduction. More precisely we assume that

$(\mathbf{H}_\Omega)$ $\mathbb{R}^N = \Omega_1 \cup \Omega_2 \cup \mathcal{H}$ with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\mathcal{H} = \partial\Omega_1 = \partial\Omega_2$ is a $W^{2,\infty}$-hypersurface in $\mathbb{R}^N$.

A consequence of this assumption is the following : if $\mathrm{d}_\mathcal{H}(\cdot)$ denotes the signed distance function to $\mathcal{H}$ which is positive in Ω_1 and negative in Ω_2, then $\mathrm{d}_\mathcal{H}$ is $W^{2,\infty}$ in a neighborhood of $\mathcal{H}$. Moreover, for $x \in \mathcal{H}$, $D\mathrm{d}_\mathcal{H}(x) = -\mathbf{n}_1(x) = \mathbf{n}_2(x)$ where, for $i = 1,2$, $\mathbf{n}_i(x)$ is the unit normal vector to $\partial\Omega_i$ pointing outwards Ω_i. We will use the notation $-\mathbf{n}_1(x)$ or $\mathbf{n}_2(x)$ for the gradient of $\mathrm{d}_\mathcal{H}$ at x, even if x does not belong to $\mathcal{H}$.

In each Ω_i $(i = 1,2)$, we have a "classical" finite-horizon control problem and the equation can be written as

$$u_t + H_i(x,t,Du) = 0 \quad \text{in } \Omega_i \times (0,T)\,, \tag{1.3}$$

for some $T > 0$, where H_i is given by

$$H_i(x,t,p) := \sup_{\alpha_i \in A_i} \{ -b_i(x,t,\alpha_i)\cdot p - l_i(x,t,\alpha_i) \}\;. \tag{1.4}$$

The b_i, l_i are at least continuous functions defined on $\overline{\Omega_i} \times (0,T) \times A_i$, the control space A_i being compact metric spaces; precise assumptions will be given later on.

Of course, one has to write down an equation on the whole space $\mathbb{R}^N$ (and in particular on $\mathcal{H}$) and this can be done using viscosity solutions' theory ([35], [5], [4]). One can consider Equation (1.1) with $H = H_i$ on Ω_i and use Ishii's definition of viscosity solutions for discontinuous Hamiltonians (cf. [28]) which reads

$$\begin{aligned} &(u^*)_t + H_*(x,t,Du^*) = 0 \quad \text{in } \mathbb{R}^N \times (0,T) \quad \text{for subsolutions } u \\ \text{and} \quad &(v_*)_t + H^*(x,t,Dv_*) = 0 \quad \text{in } \mathbb{R}^N \times (0,T) \quad \text{for supersolutions } v\,, \end{aligned}$$

where the "upper-star" denotes the upper semi-continuous envelope while the "lower-star" denotes the lower semi-continuous envelope. Following this means that we have to complement Equations (1.3) by

$$\min\{u_t + H_1(x,t,Du), u_t + H_2(x,t,Du)\} \leq 0 \quad \text{on } \mathcal{H} \times (0,T)\,, \tag{1.5}$$

$$\max\{u_t + H_1(x,t,Du), u_t + H_2(x,t,Du)\} \geq 0 \quad \text{on } \mathcal{H} \times (0,T)\,. \tag{1.6}$$

In order to present our results and to compare them with those of [6], we are going to describe the main contributions of [6] and the improvements/additional results of the present work. We first point out that the question we address in [6] (and also here) is to investigate the uniqueness properties for (1.1) or equivalently (1.3)-(1.5)-(1.6). The reason why we started to study the question in that way and why we insist on (1.3)-(1.5)-(1.6) is because of the stability properties of (1.3)-(1.5)-(1.6) : any approximation of the problem converges to a solution of (1.3)-(1.5)-(1.6) and it is, in any case, important to understand the structure of the solutions of (1.3)-(1.5)-(1.6).

The first result of [6] was to identify the maximal subsolution (and solution) and the minimal supersolution (and solution) of (1.3)-(1.5)-(1.6). Both are value functions of suitable optimal control problems and the difference between them comes from the "admissible" strategies which can be used on the interface $\mathcal{H}$ ($\mathcal{H}$ was an hyperplane in [6]). A notion of "regular" and "singular" strategies is introduced and while, for the maximal solution $\mathbf{U}^+$, only the "regular" strategies are allowed, both "regular" and "singular" strategies can be used for the minimal solution $\mathbf{U}^-$. Roughly speaking, the whole set of "regular" and "singular" strategies are those which are obtained by an approach of the dynamic and cost via differential inclusions, i.e. by using on $\mathcal{H}$ any convex combination of the dynamics and costs in Ω_1 and Ω_2. "Regular" strategies are those for which b_1 and b_2 are pointing respectively outside Ω_1 and Ω_2. The main difference between "regular" and "singular" strategies is that the "regular" ones are included in the formulation of(1.3)-(1.5)-(1.6), while this is not the case for "singular" ones.

We refer the reader to Section 2 for the description of these different control problems and in particular of the two different value functions $\mathbf{U}^-$ and $\mathbf{U}^+$, with the (classical) assumptions we are going to use. Of course, we give a precise definition of "regular" and "singular" strategies. To our point of view, there is no criterion to declare one of these value functions more natural than the other and therefore we pay the same attention to both.

In order to obtain this complete description, we have to do a double work : on one hand, we have to show that $\mathbf{U}^-$ and $\mathbf{U}^+$ are solutions of (1.3)-(1.5)-(1.6) and, maybe, to obtain additional viscosity solutions inequalities on $\mathcal{H}$. This is indeed the case for $\mathbf{U}^-$ for which taking into account "singular" strategies is translated into an additional subsolution inequality on $\mathcal{H}$, but not for $\mathbf{U}^+$, which partially justify the above sentence claiming that "regular" strategies are included in the formulation of (1.3)-(1.5)-(1.6) (see also Theorem 3.6). Then we have to study the properties of general sub and supersolutions of (1.3)-(1.5)-(1.6) and more particularly on $\mathcal{H}$. Of course, and this is rather classical, we have connect these sub and supersolutions properties with sub or super-optimality principles. This is done in Section 3.2.

The difference here with [6] is that $\mathbf{U}^-$, $\mathbf{U}^+$ are not necessarily continuous since, at the same time, we have weakened the controlability assumption and we consider finite horizon control problems. The first consequence is that the connections with the Bellman Equation (1.3)-(1.5)-(1.6) in Section 3 has to be stated in terms of discontinuous viscosity solutions (cf. Theorem 3.3). Then, still in Section 3, we provide properties, satisfied either by $\mathbf{U}^+$ or by general sub and supersolutions which play a key role in order to obtain comparison results.

The next step consists in studying uniqueness-comparison properties. Of course, there is no general comparison result for (1.3)-(1.5)-(1.6) since, in general, we have more than one solution ($\mathbf{U}^-$ and $\mathbf{U}^+$) but it turns out that, if we add a viscosity subsolution inequality on $\mathcal{H}$ (related, as we already mentioned it above, to singular strategies), then not only $\mathbf{U}^-$ becomes the only solution of this new problem but we have a full Strong Comparison Result for this new problem (i.e. a comparison result between discontinuous sub and supersolutions). This allows us to perform all the classical pde arguments in the $\mathbf{U}^-$ case. On the contrary, we were unable to find a pde characterization of $\mathbf{U}^+$ and all the proof requires optimal control arguments. This explains why we (unfortunately) have to double a lot of proofs since those for $\mathbf{U}^-$ and $\mathbf{U}^+$ have to use completely different arguments.

Compared to [6], we have modified the strategy of the comparison proofs by emphasizing the role of a "local comparison result" which is given in the Appendix. There are several reasons to do so : such local results are useful for applications, for example in homogenization problems which we consider in a forthcoming work with N. Tchou [7]; in such applications the use of the perturbed test-function of L. C. Evans [21, 22] requires (or is far more simpler with) such local comparison results. On the other hand we have to handle, at the same time, a more complex geometry than in [6] and a weaker controlability assumption (which implies that the sub solutions are not automatically Lipschitz continuous) and to argue locally allow to flatten the interface and use a double regularization procedure on the subsolutions in the tangent variables, first by sup-convolution to reduce to the Lipschitz continuous case and then by usual mollification. Here it is worth pointing out the double role of the "controlability in the normal direction" on $\mathcal{H}$: first, technically, this allows to perform the sup-convolution procedure in the tangent variables only by, roughly speaking, inducing a control of the normal derivatives of the solution by the tangent derivatives. Then the same argument implies that a subsolution which is Lipschitz continuous in the tangent variable is Lipschitz continuous with respect to all variables and this is precisely the case for the subsolution obtained by sup-convolution.

Finally, in [6], we did not really address the question of the stability properties, despite we provide few partial results. In Section 5, we study them more systematically. As we already mentioned above, the results and the proofs for $\mathbf{U}^-$ and $\mathbf{U}^+$ are completely different. For the problem satisfied by $\mathbf{U}^-$, it is (almost) a "classical" stability result proved by (almost) "classical" arguments, but contrarily to the standard results in viscosity solutions' theory, we face a difficulty because of the discontinuity on $\mathcal{H}$, difficulty which is solved in an unusual way by the controlability assumption in the normal direction. On the contrary, for the problem satisfied by $\mathbf{U}^+$, we prove the stability of controlled trajectories and costs, a rather delicate result since we have to show that the limit of trajectories with "regular" strategies is a trajectory wich can be represented by a "regular" strategy. In this second case, we have no pde approach and therefore this is the only kind of results we may hope to have.

Finally Section 6 is devoted to describe several extensions, in particular to multi-domains problems in which the domains may also depend on time.

There are more and more articles on Hamilton-Jacobi-Bellman Equations or control problems on multi-domains (also called stratified domains). We start by recalling the pioneering work by Dupuis [20] who uses similar methods to construct a numerical method for a calculus of variation problem with discontinuous integrand. Problems with a discontinuous running cost were addressed by either Garavello and Soravia [25, 26], or Camilli and Siconolfi [15] (even in an L^∞-framework) and Soravia [36]. To the best of our knowledge, all the uniqueness results use a special structure of the discontinuities as in [18, 19, 27] or an hyperbolic approach as in [3, 17]. Recent works on optimal control problem on stratified domains are the ones of Bressan and Hong [13] but also Barnard and Wolenski [10] and Rao and Zidani [31] (who mention a forthcoming work with Siconolfi [32]): in these three last works, where the approach is different since they do not start from (1.3)-(1.5)-(1.6) and instead write Bellman Equations which are adapted to the dynamic of the problem and the geometry of the discontinuities, uniqueness results are provided by a different method than ours, which completely relies on control arguments. The advantage of their methods is to allow them to handle more general stratified domains (non-smooth domains with multiple junctions) but with more restrictive controlability assumptions and without the stability results we can provide. We finally remark that problems on network (see [34], [2], [14]) share the same kind of difficulties: indeed one has to take into account the junctions as we have to deal with the interface $\mathcal{H}$.

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2 The optimal control problem

The control problem — We fix $T > 0$ and consider that, on each domain Ω_i $(i = 1, 2)$ we have a controlled dynamic given by $b_i : \overline{\Omega_i} \times [0, T] \times A_i \to \mathbb{R}^N$, where A_i is the compact metric space where the control takes its values. We have also a running cost $l_i : \overline{\Omega_i} \times [0, T] \times A_i \to \mathbb{R}$. Throughout the paper, we make the following assumption on the initial cost:

($\mathbf{H}_g$) *The function g is bounded and continuous in $\mathbb{R}^N$.*

Our main assumptions for the control problem are the following.

($\mathbf{H}_\mathrm{C}^1$) *For any $i = 1, 2$, A_i is a compact metric space and $b_i : \overline{\Omega_i} \times [0, T] \times A_i \to \mathbb{R}^N$ is a continuous bounded function. More precisely there exists $M_b > 0$, such that for any $x \in \mathbb{R}^N$, $s \in [0, T]$ and $\alpha_i \in A_i$, $i = 1, 2$,*

$$|b_i(x, s, \alpha_i)| \le M_b \; .$$

Moreover there exists $L_b \in \mathbb{R}$ such that, for any $z, z' \in \overline{\Omega_i}$, $s, s' \in [0, T]$ and $\alpha_i \in A_i$, $i = 1, 2$,

$$|b_i(z, s, \alpha_i) - b_i(z', s', \alpha_i)| \le L_b(|z - z'| + |s - s'|) \; .$$

($\mathbf{H}_{\mathrm{C}}^2$) *For any $i = 1, 2$, the function $l_i : \overline{\Omega_i} \times [0,T] \times A_i \to \mathbb{R}^N$ is a uniformly continuous, bounded function. More precisely there exists $M_l > 0$, such that for any $x \in \mathbb{R}^N$, $s \in [0,T]$ and $\alpha_i \in A_i$, $i = 1, 2$,*

$$|l_i(x, s, \alpha_i)| \leq M_l \,.$$

Moreover there exists a modulus of continuity $m_l : [0, +\infty) \to [0, +\infty)$ such that, for any $z, z' \in \overline{\Omega_i}$, $s, s' \in [0,T]$ and $\alpha_i \in A_i$, $i = 1, 2$,

$$|l_i(z, s, \alpha_i) - l_i(z', s', \alpha_i)| \leq m_l(|z - z'| + |s - s'|) \,.$$

($\mathbf{H}_{\mathrm{C}}^3$) *For each $i = 1, 2$, $z \in \overline{\Omega_i}$, and $s \in [0,T]$, the set $\big\{\big(b_i(z, s, \alpha_i), l_i(z, s, \alpha_i)\big) : \alpha_i \in A_i\big\}$ is closed and convex.*

($\mathbf{H}_{\mathrm{C}}^4$) *There is a $\delta > 0$ such that for any $i = 1, 2$, $z \in \mathcal{H}$ and $s \in [0,T]$*

$$\mathbf{B}_i(z, s) \cdot \mathbf{n}_i(z) \supset [-\delta, \delta] \tag{2.1}$$

where $\mathbf{B}_i(z, s) := \big\{b_i(z, s, \alpha_i) : \alpha_i \in A_i\big\}$.

Assumption ($\mathbf{H}_{\mathrm{C}}^1$) and ($\mathbf{H}_{\mathrm{C}}^2$) are the classical hypotheses used in control problems, while ($\mathbf{H}_{\mathrm{C}}^3$) avoids the use of relaxed controls. Hypothesis ($\mathbf{H}_{\mathrm{C}}^4$) expresses some controllability condition but only in the normal direction when the point x belongs to the boundaries shared by the sets Ω_i. In the sequel, we refer to ($\mathbf{H}_{\mathrm{C}}$) as the intersection of all the four hypotheses ($\mathbf{H}_{\mathrm{C}}^1$)–($\mathbf{H}_{\mathrm{C}}^4$).

Boundary dynamics — In order to define the controlled dynamics and trajectories which may stay for a while on the common boundary $\mathcal{H}$, we introduce the boundary dynamic as follows: if $s \in [0,T]$, $z \in \mathcal{H}$ we set

$$b_{\mathcal{H}}\big(z, s, a\big) = b_{\mathcal{H}}\big(z, s, (\alpha_1, \alpha_2, \mu)\big) := \mu b_1(z, s, \alpha_1) + (1 - \mu) b_2(z, s, \alpha_2) \,,$$

where $\mu \in [0,1]$, $\alpha_1 \in A_1$, $\alpha_2 \in A_2$. For any $z \in \mathcal{H}$ and $s \in [0,T]$ we denote by

$$A_0(z, s) := \Big\{a = (\alpha_1, \alpha_2, \mu) : b_{\mathcal{H}}\big(z, s, (\alpha_1, \alpha_2, \mu)\big) \cdot \mathbf{n}_1(z) = 0\Big\} \,,$$

and the associated cost on $\mathcal{H}$ is

$$l_{\mathcal{H}}(z, s, a) = l_{\mathcal{H}}\big(z, s, (\alpha_1, \alpha_2, \mu)\big) := \mu l_1(z, s, \alpha_1) + (1 - \mu) l_2(z, s, \alpha_2) \,.$$

Notice that the dynamic and cost on $\mathcal{H}$ are not symmetric if one swaps the indices 1 and 2 (although this could be overcome by changing also μ).

Trajectories — We are going to define the trajectories of our optimal control problem by using the approach via differential inclusions which is rather convenient here. This approach has been introduced in [37] (see also [1]) and has now become classical.

Our trajectories $X_{x,t}(\cdot) = \big((X_{x,t})_1, (X_{x,t})_2, \ldots, (X_{x,t})_N\big)(\cdot)$ are Lipschitz continuous functions which are solutions of the following differential inclusion

$$\dot{X}_{x,t}(s) \in \mathcal{B}(X_{x,t}(s), t - s) \quad \text{for a.e. } s \in [0, t) \,; \quad X_{x,t}(0) = x \tag{2.2}$$

where

$$\mathcal{B}(z,s) := \begin{cases} \mathbf{B}_i(z,s) & \text{if } z \in \Omega_i\,, \\ \overline{\mathrm{co}}\big(\mathbf{B}_1(z,s) \cup \mathbf{B}_2(z,s)\big) & \text{if } z \in \mathcal{H}\,, \end{cases} \tag{2.3}$$

the notation $\overline{\mathrm{co}}(E)$ referring to the convex closure of the set $E \subset \mathbb{R}^N$. We point out that if the definition of $\mathcal{B}(z,s)$ is natural when $z \in \Omega_i$, it is dictated by the assumptions to obtain the existence of a solution to (2.2) for $z \in \mathcal{H}$ (see below).

As we see, our controls $a(\cdot)$ can take two forms: either $a(s)$ belongs to one of the control sets A_i; or it can be expressed as a triple $(\alpha_1(s), \alpha_2(s), \mu(s)) \in A_1 \times A_2 \times [0,1]$. Hence, in order to define globally a control, we introduce the compact set

$$A := A_1 \times A_2 \times [0,1]$$

and define a control as being a function of $L^\infty(0,t;A)$ which can be seen as a subset of $\mathcal{A} := L^\infty(0,T;A)$. Let us define

$$\mathcal{E}_i := \big\{ s \in (0,t) : X_{x,t}(s) \in \Omega_i \big\}\,, \quad \mathcal{E}_\mathcal{H} := \big\{ s \in (0,t) : X_{x,t}(s) \in \mathcal{H} \big\}\,,$$

where actually these sets depend on (x,t) but we shall omit this dependence for the sake of simplicity of notations. We then have the following

Theorem 2.1. *Assume* $(\mathbf{H}_\Omega)$, $(\mathbf{H}_\mathrm{C}^1)$, $(\mathbf{H}_\mathrm{C}^2)$ *and* $(\mathbf{H}_\mathrm{C}^3)$. *Then*

(i) *For each* $x \in \mathbb{R}^N$, $t \in [0,T)$ *there exists a Lipschitz function* $X_{x,t} : [0,t] \to \mathbb{R}^N$ *which is a solution of the differential inclusion* (2.2).

(ii) *For each solution* $X_{x,t}(\cdot)$ *of* (2.2), *there exists a control* $a(\cdot) \in \mathcal{A}$ *such that for a.e.* $s \in (0,t)$

$$\dot{X}_{x,t}(s) = \sum_{i=1,2} b_i\big(X_{x,t}(s), t-s, \alpha_i(s)\big)\mathbb{1}_{\mathcal{E}_i}(s) + b_\mathcal{H}\big(X_{x,t}(s), t-s, a(s)\big)\mathbb{1}_{\mathcal{E}_\mathcal{H}}(s) \tag{2.4}$$

where $a(s) = \big(\alpha_1(s), \alpha_2(s), \mu(s)\big)$ *if* $X_{x,t}(s) \in \mathcal{H}$.

(iii) *If* $\mathbf{e}(\cdot) = \mathbf{n}_1(\cdot)$ *or* $\mathbf{n}_2(\cdot)$ *we have*

$$b_\mathcal{H}\big(X_{x,t}(s), t-s, a(s)\big) \cdot \mathbf{e}\big(X_{x,t}(s)\big) = 0 \quad \text{for a.e. } s \in \mathcal{E}_\mathcal{H}\,.$$

In other words, $a(s) \in A_0(X_{x,t}(s), t-s)$ *for a.e.* $s \in \mathcal{E}_\mathcal{H}$.

Proof. The proof is done exactly as in [6], the only minor modification consisting in adding the time variable in the vector field b. □

Regular and Singular dynamics — It is worth remarking that, in Theorem 2.1, a solution $X_{x,t}(\cdot)$ can be associated to several controls $a(\cdot)$. So, to properly set the control problem we introduce the set $\mathcal{T}_{x,t}$ of admissible controlled trajectories starting from x,

$$\mathcal{T}_{x,t} := \Big\{ \big(X_{x,t}(\cdot), a(\cdot)\big) \in \mathrm{Lip}(0,t;\mathbb{R}^N) \times \mathcal{A} \text{ such that (2.4) is fulfilled and } X_{x,t}(0) = x \Big\}\,.$$

Given $(z,s) \in \mathcal{H} \times [0,t]$, we call *singular* a dynamic $b_\mathcal{H}(z,s,a)$ with $a = (\alpha_1, \alpha_2, \mu) \in A_0(z,s)$ when

$$b_1(z,s,\alpha_1) \cdot \mathbf{n}_1(z) < 0\,, \quad b_2(z,s,\alpha_2) \cdot \mathbf{n}_2(z) < 0\,.$$

Conversely, the *regular* dynamics are those for which the $b_i(z,s,\alpha_i) \cdot \mathbf{n}_i(z) \geq 0$ $(i = 1,2)$. The set of regular controls is denoted by

$$A_0^{\mathrm{reg}}(z,s) := \big\{ a = (\alpha_1, \alpha_2, \mu) \in A_0(z,s)\,;\; b_i(z,s,\alpha_i) \cdot \mathbf{n}_i(z) \geq 0,\; i = 1,2 \big\}\,,$$

and the regular trajectories are defined as

$$\mathcal{T}_{x,t}^{\mathrm{reg}} := \Big\{ \big(X_{x,t}(\cdot), a(\cdot)\big) \in \mathcal{T}_{x,t} : \text{ for a.e. } s \in \mathcal{E}_\mathcal{H},\; a(s) \in A_0^{\mathrm{reg}}\big(X(s), t-s\big) \Big\}\,.$$

Trajectories satisfying for example $b_1(z,s,\alpha_1) \cdot \mathbf{n}_1(z) < 0 < b_2(z,s,\alpha_2) \cdot \mathbf{n}_2(z)$ are neither called singular nor regular since they do not remain on $\mathcal{H}$, they are handled by classical arguments.

THE COST FUNCTIONAL – Our aim is to minimize a finite horizon cost functional such that we respectively pay l_i if the trajectory is in Ω_i, and $l_\mathcal{H}$ if it is on $\mathcal{H}$. The final cost is given by g.

More precisely, the cost associated to $(X_{x,t}(\cdot), a) \in \mathcal{T}_{x,t}$ is

$$J(x,t;(X_{x,t},a)) := \int_0^t \ell\big(X_{x,t}(s), t-s, a(s)\big)\,\mathrm{d}s + g\big(X_{x,t}(t)\big) \tag{2.5}$$

where the Lagrangian is given by

$$\ell\big(X_{x,t}(s), t-s, a(s)\big) := \sum_{i=1,2} l_i\big(X_{x,t}(s), t-s, \alpha_i(s)\big)\mathbb{1}_{\mathcal{E}_i}(s) + l_\mathcal{H}\big(X_{x,t}(s), t-s, a(s)\big)\mathbb{1}_{\mathcal{E}_\mathcal{H}}(s)\,. \tag{2.6}$$

THE VALUE FUNCTIONS – For each $x \in \mathbb{R}^N$ and $t \in [0,T)$, we define the following two value functions

$$\mathbf{U}^-(x,t) := \inf_{(X_{x,t},a) \in \mathcal{T}_{x,t}} J\big(x,t;(X_{x,t},a)\big) \tag{2.7}$$

$$\mathbf{U}^+(x,t) := \inf_{(X_{x,t},a) \in \mathcal{T}_{x,t}^{\mathrm{reg}}} J\big(x,t;(X_{x,t},a)\big). \tag{2.8}$$

A first key result is the **Dynamic Programming Principle** (the proof being standard once we have the definition of trajectories, we skip it).

Theorem 2.2. *Assume* $(\mathbf{H}_\Omega)$, $(\mathbf{H}_\mathrm{C}^1)$, $(\mathbf{H}_\mathrm{C}^2)$ *and* $(\mathbf{H}_\mathrm{C}^3)$. *Let* $\mathbf{U}^-, \mathbf{U}^+$ *be the value functions defined in* (2.7) *and* (2.8). *Then for each* $(x,t) \in \mathbb{R}^N \times [0,T)$, *and each* $\tau \in (0,t)$, *we have*

$$\mathbf{U}^-(x,t) = \inf_{(X_{x,t},a) \in \mathcal{T}_{x,t}} \left\{ \int_0^\tau \ell\big(X_{x,t}(s), t-s, a(s)\big)\,\mathrm{d}s + \mathbf{U}^-(X_{x,t}(\tau), t-\tau) \right\} \tag{2.9}$$

$$\mathbf{U}^+(x,t) = \inf_{(X_{x,t},a) \in \mathcal{T}_{x,t}^{\mathrm{reg}}} \left\{ \int_0^\tau \ell\big(X_{x,t}(s), t-s, a(s)\big)\,\mathrm{d}s + \mathbf{U}^+(X_{x,t}(\tau), t-\tau) \right\}. \tag{2.10}$$

We will prove that both value functions are continuous, but here it is not so immediate since we only assume controlability in the normal directions. We postpone this proof which uses some comparison for the semi-continuous envelopes.

3 The pde formulation of the problem

In order to describe what is happening on the hypersurface $\mathcal{H}$, we shall introduce two "tangential Hamiltonians", namely H_T, H_T^{reg}. We introduce some notations to be clear on how they are defined.

We shall consider the tangent bundle $T\mathcal{H} := \cup_{z\in\mathcal{H}}\big(\{z\}\times T_z\mathcal{H}\big)$ where $T_z\mathcal{H}$ is the tangent space to $\mathcal{H}$ at z (which is essentially $\mathbb{R}^{N-1}$). Thus, if $\phi\in C^1(\mathcal{H})$, and $x\in\mathcal{H}$, we denote by $D_\mathcal{H}\phi(x)$ the gradient of ϕ at x, which belongs to $T_x\mathcal{H}$.

Also, the scalar product in $T_z\mathcal{H}$ will be denoted by $\langle u,v\rangle$ (we drop the reference to $T_z\mathcal{H}$ for simplicity, since no confusion has to be feared in the sequel). In this definition, both vectors u,v should belong to $T_z\mathcal{H}$ for this definition to make sense. Hence, to be precise we should use the orthogonal projection $P_z:\mathbb{R}^N\to T_z\mathcal{H}$ when at least one of the vectors u,v lives in $\mathbb{R}^N$, but we shall omit this point when writing $\big\langle b_\mathcal{H}(x,t,a), D_\mathcal{H}\phi(x,t)\big\rangle$. Indeed, for any control a in $A_0(x,t)$ or $A_0^{\text{reg}}(x,t)$, $b_\mathcal{H}(x,t,a)$ can be identified with $P_x b_\mathcal{H}(x,t,a)$ since $b_\mathcal{H}(x,t,a)$ has no component on the normal direction to $\mathcal{H}$, by definition. To avoid confusions, the notation $u\cdot v$ will refer only to the usual Euclidian scalar product in $\mathbb{R}^N$.

The Hamiltonians H_T, H_T^{reg} will be written as $H_T/H_T^{\text{reg}}(x,t,p)$ where $((x,p),t)\in T\mathcal{H}\times[0,T]$. They are defined as follows:

$$H_T(x,t,p) := \sup_{A_0(x,t)}\big\{-\big\langle b_\mathcal{H}(x,t,a),p\big\rangle - l_\mathcal{H}(x,t,a)\big\}\,, \tag{3.1}$$

$$H_T^{\text{reg}}(x,t,p) := \sup_{A_0^{\text{reg}}(x,t)}\big\{-\big\langle b_\mathcal{H}(x,t,a),p\big\rangle - l_\mathcal{H}(x,t,a)\big\}\,, \tag{3.2}$$

where $A_0(x,t), A_0^{\text{reg}}(x,t)$ have been defined above.

The definition of viscosity sub and super-solutions for H_T and H_T^{reg} have to be understood on $\mathcal{H}$ as follows:

Definition 3.1. *A bounded usc function $u:\mathcal{H}\times[0,T]\to\mathbb{R}$ is a viscosity subsolution of*

$$u_t(x,t)+H_T(x,t,D_\mathcal{H}u)=0 \quad \textit{on} \quad \mathcal{H}\times[0,T]$$

if, for any $\phi\in C^1(\mathcal{H}\times[0,T])$ and any maximum point (x,t) of $(z,s)\mapsto u(z,s)-\phi(z,s)$ in $\mathcal{H}\times[0,T]$, one has

$$\phi_t(x,t)+H_T\big(x,t,D_\mathcal{H}\phi(x,t)\big)\le 0\,.$$

Notice that of course, $(x,D_\mathcal{H}\phi(x,t))\in T\mathcal{H}$, so that this is coherent with the definition of H_T. A similar definition holds for H_T^{reg}, for supersolutions and solutions. Of course, if u is defined in a bigger set containing $\mathcal{H}\times[0,T]$ (typically $\mathbb{R}^N\times[0,T]$), we have to use $u|_{\mathcal{H}\times[0,T]}$ (the restriction of u to $\mathcal{H}\times[0,T]$) in this definition, a notation that we will omit when not necessary.

For the sake of clarity we introduce now a global formulation involving a complementary Hamiltonian on the interface $\mathcal{H}$. To begin with, we recall that a subsolution (resp. a supersolution) of (1.1) when $H(x,t,p)=H_1(x,t,p)$ if $x\in\Omega_1$ and $H(x,t,p)=H_2(x,t,p)$ if $x\in\Omega_2$ is a bounded usc function u (resp.a bounded lsc function v) which satisfies

$$\begin{cases} u_t + H_1(x,t,Du) \leq 0 & \text{in } \Omega_1 \times (0,T)\,, \\ u_t + H_2(x,t,Du) \leq 0 & \text{in } \Omega_2 \times (0,T)\,, \\ u_t + \min\{H_1(x,t,Du), H_2(x,t,Du)\} \leq 0 & \text{in } \mathcal{H} \times (0,T)\,, \end{cases} \tag{3.3}$$

$$\left[\quad \text{resp.} \quad \begin{cases} v_t + H_1(x,t,Dv) \geq 0 & \text{in } \Omega_1 \times (0,T)\,, \\ v_t + H_2(x,t,Dv) \geq 0 & \text{in } \Omega_2 \times (0,T)\,, \\ v_t + \max\{H_1(x,t,Dv), H_2(x,t,Dv)\} \geq 0 & \text{in } \mathcal{H} \times (0,T) \end{cases} \right]. \tag{3.4}$$

Recall that since each b_i is defined on $\overline{\Omega_i} \times (0,T) \times \mathbb{R}$, then H_i is well-defined on $\mathcal{H} \times (0,T)$. Next we have the following definition.

Definition 3.2. *We say that a bounded usc function u is a* <u>subsolution</u> *of*

$$u_t + \mathbb{H}^-(x,t,Du) = 0 \text{ in } \mathbb{R}^N \times (0,T) \tag{3.5}$$

$$\left[\quad \textit{resp.} \quad u_t + \mathbb{H}^+(x,t,Du) = 0 \text{ in } \mathbb{R}^N \times (0,T) \quad \right] \tag{3.6}$$

if it satisfies (3.3) *and*

$$u_t(x,t) + H_T(x,t,D_{\mathcal{H}}u) \leq 0 \quad \text{on} \quad \mathcal{H} \times [0,T]\,,$$

$$\left[\quad \textit{resp.} \quad u_t(x,t) + H_T^{\mathrm{reg}}(x,t,D_{\mathcal{H}}u) \leq 0 \quad \text{on} \quad \mathcal{H} \times [0,T]\,, \quad \right]$$

in the sense of Definition 3.1.
A lsc function v is a <u>supersolution</u> *of* (3.5) *or* (3.6) *if it satisfies* (3.4).

Notice that in this definition, a complementary condition is required only for the subsolution, nothing more is added for the supersolution.

3.1 Properties of $\mathbf{U}^+$ and $\mathbf{U}^-$

We shall prove later on that both $\mathbf{U}^+$ and $\mathbf{U}^-$ are continuous, but for the moment we have to treat them a priori as discontinuous viscosity solutions of some problem. We recall that, for any bounded function v, the lower and upper semi-continuous envelopes are defined by

$$v_*(x,t) := \liminf_{(z,s)\to(x,t)} v(z,s)\,, \quad v^*(x,t) := \limsup_{(z,s)\to(x,t)} v(z,s)\,.$$

Then, as we mention in the introduction the definition of viscosity solution for discontinuous solutions is modified by taking $(\mathbf{U}^-)_*$ instead of $\mathbf{U}^-$ for the supersolution condition, and $(\mathbf{U}^-)^*$ instead of $(\mathbf{U}^-)$ for the subsolution condition.

We claim that the value functions $\mathbf{U}^-$ and $\mathbf{U}^+$ are viscosity solutions of the Hamilton-Jacobi-Bellman problem (1.3)-(1.5)-(1.6), while they fulfill different inequalities on the hyperplane $\mathcal{H}$.

Theorem 3.3. *Assume* $(\mathbf{H}_g)$, $(\mathbf{H}_\Omega)$ *and* $(\mathbf{H}_\mathrm{C})$. *Then value functions* $\mathbf{U}^-$ *and* $\mathbf{U}^+$ *are both viscosity solutions of* $u_t + H(x,u,Du) = 0$. *Moreover,* $\mathbf{U}^-$ *is a subsolution of* $u_t + \mathbb{H}^-(x,t,Du) = 0$ *while* $\mathbf{U}^+$ *is a subsolution of* $u_t + \mathbb{H}^+(x,t,Du) = 0$.

Proof. The proof follows the arguments of [6, Thm 2.5] with some adaptations due to the fact that $\mathbf{U}^-, \mathbf{U}^+$ can be discontinuous. We briefly show how to adapt the arguments. In order to prove that $(\mathbf{U}^-)_*$ is a supersolution we consider a point (x,t) where $(\mathbf{U}^-)_* - \phi$ reaches its minimum, ϕ being a smooth test function. If x belongs to some Ω_i, the proof is classical since everything can be done in Ω_i around the time t.

Thus we assume that $x \in \mathcal{H}$ and that the minimum is strict in $B(x,r) \times (t-\sigma, t+\sigma)$ for some $r, \sigma > 0$. There exists a sequence $(x_n, t_n) \in B(x,r) \times (t-\sigma, t+\sigma)$ which converges to (x,t) such that $\mathbf{U}^-(x_n,t_n) \to (\mathbf{U}^-)_*(x,t)$ and by the dynamic programming principle,

$$\mathbf{U}^-(x_n,t_n) = \inf_{(X_{x_n,t_n},a)\in\mathcal{T}_{x_n,t_n}} \left\{ \int_0^\tau \ell\big(X_{x_n,t_n}(s), t_n - s, a(s)\big)\,\mathrm{d}s + \mathbf{U}^-\big(X_{x_n,t_n}(\tau), t_n-\tau\big) \right\},$$

where $\tau < \sigma$. Using that (i) $\mathbf{U}^-(x_n,t_n) = (\mathbf{U}^-)_*(x,t) + o_n(1)$ where $o_n(1) \to 0$, (ii) $\mathbf{U}^-\big(X_{x_n,t_n}(\tau), t_n - \tau\big) \geq \mathbf{U}^-_*\big(X_{x_n,t_n}(\tau), t_n-\tau\big)$ and the maximum point property, we obtain

$$\phi(x_n,t_n) + o_n(1) \geq \inf_{(X_{x_n,t_n},a)\in\mathcal{T}_{x_n,t_n}} \left\{ \int_0^\tau \ell\big(X_{x_n,t_n}(s), t_n - s, a(s)\big)\,\mathrm{d}s + \phi\big(X_{x_n,t_n}(\tau), t_n-\tau\big) \right\}.$$

Now we use the expansion of $\phi(X_{x_n,t_n}(\tau), t_n-\tau)$, and noting $X(\cdot) = X_{x_n,t_n}(\cdot)$ for simplicity, we rewrite the inequality as $o_n(1) \leq \sup_{(X,a)} \int_0^\tau \delta[\phi](s)\,\mathrm{d}s$ where

$$\begin{aligned}
&\delta[\phi](s) := \\
&\Big(-l_1(X(s),t_n-s,\alpha_1(s)) - b_1(X(s),t_n-s,\alpha_1(s))\cdot D\phi(X(s),t_n-s) + \phi_t(X(s),t_n-s)\Big)\mathbb{1}_{\mathcal{E}_1}(s)\\
&+\Big(-l_2(X(s),t_n-s,\alpha_2(s)) - b_2(X(s),t_n-s,\alpha_2(s))\cdot D\phi(X(s),t_n-s) + \phi_t(X(s),t_n-s)\Big)\mathbb{1}_{\mathcal{E}_2}(s)\\
&+\Big(-l_{\mathcal{H}}(X(s),t_n-s,a(s)) - b_{\mathcal{H}}(X(s),t_n-s,a(s))\cdot D\phi(X(s),t_n-s) + \phi_t(X(s),t_n-s)\Big)\mathbb{1}_{\mathcal{E}_H}(s)\\
&\leq \Big(\phi_t(X(s),t_n-s) + H_1\Big(X(s),t_n-s,D\phi(X(s),t_n-s)\Big)\Big)\mathbb{1}_{\mathcal{E}_1}(s)\\
&+\Big(\phi_t(X(s),t_n-s) + H_2\Big(X(s),t_n-s,D\phi(X(s),t_n-s)\Big)\Big)\mathbb{1}_{\mathcal{E}_2}(s)\\
&+\Big(\phi_t(X(s),t_n-s) + H_T\Big(X(s),t_n-s,D\phi(X(s),t_n-s)\Big)\Big)\mathbb{1}_{\mathcal{E}_H}(s).
\end{aligned}$$

Using that $H_1, H_2, H_T \leq \max(H_1,H_2)$ (only on $\mathcal{H}$ for H_T), letting $n\to\infty$ and then dividing by τ and sending τ to zero, we obtain

$$\max\big(\phi_t + H_1, \phi_t + H_2\big)\big(x,t,D\phi(x,t)\big) \geq 0\,,$$

which is the viscosity supersolution condition. The proof for $(\mathbf{U}^+)_*$ is exactly the same, with H_T replaced by H_T^{reg}, which satisfies also $H_T^{\mathrm{reg}} \leq \max(H_1,H_2)$ on $\mathcal{H}$.

For the subsolution condition, we have to consider maximum points of $(\mathbf{U}^-)^* - \phi$, ϕ being again a smooth function. If such maximum point are in Ω_1 or Ω_2, the proof is again classical. Hence we consider the case when $(\mathbf{U}^-)^* - \phi$ reaches a strict local maximum at (x,t) with $x \in \mathcal{H}$, $t \in (0,T)$.

Then there exists a sequence $(x_n,t_n) \to (x,t)$ such that $\mathbf{U}^-(x_n,t_n) \to (\mathbf{U}^-)^*(x,t)$ and our first claim is that we can assume that $x_n \in \mathcal{H}$. Indeed, if $x_n \in \Omega_1$, we use assumption $(\mathbf{H}_C^4)$: there exists α_i such that $b_1(x,t,\alpha_1) \cdot \mathbf{n}_1(x) = \delta$. Considering the trajectory with the constant control α_1

$$\dot{Y}(s) = b_1(Y(s), t_n - s, \alpha_1) \quad , \quad Y(0) = x_n,$$

it is easy to show that τ_n^1, the first exit time of the trajectory Y from Ω_1 tends to 0 as $n \to +\infty$. By the Dynamic Programming Principle, denoting $(\tilde{x}_n, \tilde{t}_n) = (X(\tau_n^1), t - \tau_n^1)$, we have

$$\mathbf{U}^-(x_n,t_n) \leq \int_0^{\tau_n^1} \ell\big(Y(s), t_n - s, \alpha_1\big)\,\mathrm{d}s + \mathbf{U}^-(\tilde{x}_n, \tilde{t}_n) = \mathbf{U}^-(\tilde{x}_n, \tilde{t}_n) + o_n(1),$$

where $o_n(1) \to 0$. Therefore $\mathbf{U}^-(\tilde{x}_n, \tilde{t}_n) \to (\mathbf{U}^-)^*(x,t)$ and $\tilde{x}_n \in \mathcal{H}$.

Assuming that $x_n \in \mathcal{H}$, we can use again the Dynamic Programming Principle

$$\mathbf{U}^-(x_n,t_n) \leq \int_0^{\tau} \ell\big(X_{x_n,t_n}(s), t_n - s, a(s)\big)\,\mathrm{d}s + \mathbf{U}^-\big(X_{x_n,t_n}(\tau), t_n - \tau\big),$$

with constant controls $a(s) = \alpha_i$ with $b_i(x,t,\alpha_i) \cdot \mathbf{n}_i(x) < 0$. Arguing as above we get

$$\phi_t(x,t) - b_i(x,t,\alpha_i) \cdot D\phi(x,t) - l_i(x,t,\alpha_i) \leq 0 \, .$$

Moreover, combining Assumptions $(\mathbf{H}_\mathrm{C}^3)$ and $(\mathbf{H}_\mathrm{C}^4)$, one proves easily that this inequality holds for any α_i with $b_i(x,t,\alpha_i) \cdot \mathbf{n}_i(x) \leq 0$.

Taking these informations into account, if we assume by contradiction that

$$\min\big\{\phi_t(x,t) + H_1\big(x,t,D\phi(x,t)\big) \, ; \, \phi_t(x,t) + H_2\big(x,t,D\phi(x,t)\big)\big\} > 0 \, ,$$

this means that there exists α_1, α_2 with if $b_1(x,t,\alpha_1) \cdot \mathbf{n}_1(x) > 0$ and $b_2(x,t,\alpha_2) \cdot \mathbf{n}_2(x) > 0$ such that, for $i = 1,2$

$$\phi_t(x,t) - b_i(x,t,\alpha_i) \cdot D\phi(x,t) - l_i(x,t,\alpha_i) > 0 \, .$$

For (y,s) close to (x,t) and for such α_1, α_2, we set

$$\mu^\sharp(y,s) := \frac{b_2(y,s,\alpha_2) \cdot \mathbf{n}_2(y)}{b_1(y,s,\alpha_1) \cdot \mathbf{n}_1(y) + b_2(y,s,\alpha_2) \cdot \mathbf{n}_2(y)} \, .$$

Then we solve the ode

$$\dot{x}(s) = \mu^\sharp(x(s), t-s) b_1(x(s), t-s, \alpha_1) + (1 - \mu^\sharp(x(s), t-s)) b_2(x(s), t-s, \alpha_2) \, , \qquad x(0) = x.$$

By our hypotheses on b_1 and b_2, the right-hand side is Lipschitz continuous so that the Cauchy-Lipschitz applies and gives a solution $x(s)$. Moreover, by our choice of $\mu^\sharp$, it is clear that $0 \leq \mu^\sharp \leq 1$ and that $\dot{x}(s) \cdot \mathbf{n}_1(x(s)) = 0$, which implies by Gronwall's lemma that $s \mapsto x(s)$ remains on $\mathcal{H}$, at least until some time $\tau > 0$. Using again the Dynamic Programming Principle and the usual arguments, we are lead to

$$\mu^\sharp(x,t)\Big(\phi_t(x,t) - b_1(x,t,\alpha_1)\cdot D\phi(x,t) - l_1(x,t,\alpha_1)\Big)$$
$$+ (1-\mu^\sharp(x,t))\Big(\phi_t(x,t) - b_2(x,t,\alpha_2)\cdot D\phi(x,t) - l_2(x,t,\alpha_2)\Big) \leq 0\ ,$$

a contradiction.

Finally the H_T-inequality follows from the same arguments : in particular, if $b_1(x,t,\alpha_1)\cdot\mathbf{n}_1(x) < 0$ and $b_2(x,t,\alpha_1)\cdot\mathbf{n}_2(x) < 0$, the above $\mu^\sharp$-argument can be applied readily.

The same proof works also for $(\mathbf{U}^+)^*$, except that some situation cannot occur since we are only considering regular dynamics. $\square$

Our next result is a (little bit unusual) supersolution property which is satisfied by $\mathbf{U}^+$ on $\mathcal{H}$, which is done exactly as in of [6, Thm 2.7] once we have the following extension result

Lemma 3.4. *Let us assume that* $(\mathbf{H}_\Omega)$ *holds and let* $\phi \in C^1\big(\mathcal{H}\times[0,T]\big)$. *Then there exists a function* $\tilde\phi \in C^1\big(\mathbb{R}^N\times[0,T]\big)$ *such that* $\tilde\phi = \phi$ *in* $\mathcal{H}\times[0,T]$.

Proof. The proof is rather classical so that we omit it. $\square$

We are going to consider control problems set in either Ω_i or its closure. For the sake of clarity we use the following notation. If $x\in\Omega_i$, and $\alpha_i(\cdot)\in L^\infty([0,T];A_i)$, we will denote by $Y^i_{x,t}(\cdot)$ the solution of the following ode

$$\dot Y^i_{x,t}(s) = b_i(Y^i_{x,t}(s), t-s, \alpha_i(s)) \quad , \quad Y^i_{x,t}(0) = x\,. \tag{3.7}$$

The following result is playing a key role in order to prove that the value function $\mathbf{U}^+$ is continuous and the maximal subsolution of (1.3)-(1.5)-(1.6)-(4.3) (see Theorem 4.4 below). One of the key difference between the $\mathbf{U}^-$ and $\mathbf{U}^+$ cases is that for the $\mathbf{U}^+/H_T^{\mathrm{reg}}$ case, we are able to prove such result only for the supersolution $(\mathbf{U}^+)_*$, while, in the other case $(\mathbf{U}^-/H_T)$, it is true for any supersolution (see Theorem 3.8 below).

Theorem 3.5. *Assume* $(\mathbf{H}_g)$, $(\mathbf{H}_\Omega)$ *and* $(\mathbf{H}_\mathrm{C})$. *Let* $\phi\in C^1\big(\mathcal{H}\times[0,T]\big)$ *and suppose that* (x,t) *is a minimum point of* $(z,s)\mapsto(\mathbf{U}^+)_*(z,s)-\phi(z,s)$ *in* $\mathcal{H}\times[0,T]$. *Then we have either*

A) *there exist* $\eta>0$, $i\in\{1,2\}$ *and a control* $\alpha_i(\cdot)$ *such that,* $Y^i_{x,t}(s)\in\Omega_i$ *for all* $s\in]0,\eta]$ *and*

$$(\mathbf{U}^+)_*(x,t) \geq \int_0^\eta l_i(Y^i_{x,t}(s), t-s, \alpha_i(s))\,\mathrm{d}s + (\mathbf{U}^+)_*(Y^i_{x,t}(\eta), t-\eta) \tag{3.8}$$

or

B) *it holds*

$$\partial_t\phi(x,t) + H_T^{\mathrm{reg}}\big(x,t,D_\mathcal{H}\phi(x,t)\big) \geq 0. \tag{3.9}$$

Proof. Since $x \in \mathcal{H}$, by assumption $(\mathbf{H}_{\mathrm{C}}^3)$, there exists a regular optimal control $a(\cdot) \in \mathcal{T}_{x,t}^{\mathrm{reg}}$ such that

$$\mathbf{U}^+(x,t) = \int_0^t \ell\big(X_{x,t}(s), t-s, a(s)\big)\,\mathrm{d}s + g(X_{x,t}(t))\ .$$

Moreover, by the Dynamic Programming Principle, we have, for any $\tau > 0$

$$\mathbf{U}^+(x,t) = \int_0^\tau \ell\big(X_{x,t}(s), t-s, a(s)\big)\,\mathrm{d}s + \mathbf{U}^+(X_{x,t}(\tau), t-\tau)\ .$$

We argue depending on whether or not there exists a sequence $(\tau_k)_k$ converging to 0 such that $\tau_k > 0$ and $X_{x,t}(\tau_k) \in \mathcal{H}$.

If it is NOT the case then this means that we are in the case **A)** since, for η small enough, the trajectory $X_{x,t}(\cdot)$ stays necessarily either in Ω_1 or in Ω_2 on $]0,\eta]$. Therefore we can assume for instance that $X_{x,t}(\cdot) = Y^i_{x,t}(\cdot)$ and take $\tau = \eta$ in the above equality.

On the contrary, if IT IS the case, we can use the minimum point property: assuming without loss of generality that $\phi(x,t) = (\mathbf{U}^+)_*(x,t)$, we extend ϕ to $\mathbb{R}^N \times [0,T]$ thanks to Lemma 3.4 and write, for k large enough,

$$\tilde{\phi}(x,t) \geq \int_0^{\tau_k} \ell\big(X_{x,t}(s), t-s, a(s)\big)\,\mathrm{d}s + \tilde{\phi}(X_{x,t}(\tau_k), t-\tau_k)\ .$$

The rest of the proof is the same as [6, Thm 2.7]: we obtain a contradiction by assuming

$$\phi_t(x,t) + H_T^{\mathrm{reg}}\big(x,t,D_{\mathcal{H}}\phi(x,t)\big) \leq -\eta < 0\,,$$

using the normal controllability condition $(\mathbf{H}_{\mathrm{C}}^4)$ instead of the more general (and usual) one which was used in [6]. $\square$

3.2 Properties of sub and supersolutions

Theorem 3.6. *Assume* $(\mathbf{H}_\Omega)$ *and* $(\mathbf{H}_{\mathrm{C}})$*. If* $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ *is a bounded viscosity subsolution of* $u_t + H(x,t,Du) = 0$*, then* u *is a subsolution of* $u_t + \mathbb{H}^+(x,t,Du) = 0$*.*

Proof. It is enough to check the subsolution condition only on $\mathcal{H}$ since the property clearly holds in each Ω_i by definition.

We recall that $u^*|_{\mathcal{H}\times[0,T]}$ is the restriction of u^* to $\mathcal{H} \times [0,T]$. Let $\phi(\cdot)$ be a C^1-function on $\mathcal{H}$ and $(\bar{x},\bar{t})$ a maximum point of $u^*|_{\mathcal{H}\times[0,T]} - \phi$ on $\mathcal{H} \times [0,T]$. Our aim is then to prove that, for any $a \in \mathcal{A}_0^{\mathrm{reg}}(\bar{x},\bar{t})$ we have

$$\phi_t(\bar{x},\bar{t}) - \big\langle b_{\mathcal{H}}\big(\bar{x},\bar{t},a\big), D_{\mathcal{H}}\phi(\bar{x},\bar{t})\big\rangle - l_{\mathcal{H}}\big(\bar{x},\bar{t},a\big) \leq 0\,. \tag{3.10}$$

This proof follows [6, Thm. 3.1] so that we only mention here the modifications. First, we extend ϕ by $\tilde{\phi}$ given by Lemma 3.4. Then for $\varepsilon \ll 1$ and $(z,s) \in \mathcal{H} \times [0,T]$ we consider the function

$$(z,s) \mapsto u(z,s) - \tilde{\phi}(z,s) - \eta\,\mathrm{d}_{\mathcal{H}}(z) - \frac{\mathrm{d}_{\mathcal{H}}(z)^2}{\varepsilon^2} - |z-x|^2 - |s-t| := u(z,s) - \psi_\varepsilon(z,s)\,, \tag{3.11}$$

where we recall that $\mathrm{d}_{\mathcal{H}}(\cdot)$ is the signed distance function to $\mathcal{H}$ which is positive in Ω_1 and negative in Ω_2.

Writing $a = (\alpha_1, \alpha_2, \mu)$, we assume that we are in the situation when $b_1(\bar{x}, \bar{t}, \alpha_1) \cdot \mathbf{n}_1(\bar{x}) < 0$ (and the same for index 2), since the case of non-strict inequalities can be recovered by hypothesis $(\mathbf{H}_{\mathrm{C}}^4)$ as in Thm. 3.3 (recall that a being a regular control, the opposite signs are forbidden). We choose $\eta > \bar{\eta}$ where $\bar{\eta}$ is a solution of the following equation (which has a solution under the assumption above of strict signs):

$$\tilde{\phi}_t(\bar{x}, \bar{t}) - b_1(\bar{x}, \bar{t}, \alpha_1) \cdot \big(D\tilde{\phi}(\bar{x}, \bar{t}) + \bar{\eta}\mathbf{n}_2(\bar{x})\big) - l_1(\bar{x}, \bar{t}, \alpha_1) = 0\,.$$

The rest of the proof follows the cited reference: thanks to the penalization terms, for ε small enough, $u^* - \psi_\varepsilon$ reaches its max at some point $(x_\varepsilon, t_\varepsilon) \in \overline{\Omega_2} \times [0, T]$. Then, using the equation in $\Omega_2 \times [0, T]$ or on $\mathcal{H} \times [0, T]$ leads to

$$\tilde{\phi}_t(\bar{x}, \bar{t}) - b_2(\bar{x}, \bar{t}, \alpha_2) \cdot \big(D\tilde{\phi}(\bar{x}, \bar{t}) + \eta\mathbf{n}_2(\bar{x})\big) - l_2(\bar{x}, \bar{t}, \alpha_2) \leq o_\varepsilon(1)\,.$$

We let ε tend to zero first, and then η to $\bar{\eta}$. Using the specific value of $\bar{\eta}$ leads to

$$\tilde{\phi}_t(\bar{x}, \bar{t}) - b_{\mathcal{H}}(\bar{x}, \bar{t}, a) \cdot D\tilde{\phi}(\bar{x}, \bar{t}) - l_{\mathcal{H}}(\bar{x}, \bar{t}, a) \leq 0\,,$$

that we interpret as (3.10) since $b_{\mathcal{H}}(\bar{x}, \bar{t}, a)$ has no component on the normal direction to $\mathcal{H}$ and by construction, $D_{\mathcal{H}}(\tilde{\phi}|_{\mathcal{H}}) = D_{\mathcal{H}}\phi$. □

The following lemma states a super and a sub optimality principle respectively for super and subsolutions of $w_t + H(x, t, Dw) = 0$. The proof is classical (see [9, 11, 12] and also the proof of [6, Lem. 3.2]).

Lemma 3.7. *Assume* $(\mathbf{H}_\Omega)$ *and* $(\mathbf{H}_{\mathrm{C}})$. *Let* $v : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ *be a lsc supersolution of* $v_t + H(x, t, Dv) = 0$ *and* $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ *be a usc subsolution of* $u_t + H(x, t, Du) = 0$. *Then, if* $x \in \Omega_i$ *(*$i \in \{1, 2\}$*), we have for all* $\sigma \in [0, t]$

$$v(x, t) \geq \inf_{\alpha_i(\cdot), \theta_i} \left[\int_0^{\sigma \wedge \theta_i} l_i\big(Y^i_{x,t}(s), t - s, \alpha_i(s)\big)\,\mathrm{d}s + v\big(Y^i_{x,t}(\sigma \wedge \theta_i), t - (\sigma \wedge \theta_i)\big) \right], \tag{3.12}$$

and

$$u(x, t) \leq \inf_{\alpha_i(\cdot)} \sup_{\theta_i} \left[\int_0^{\sigma \wedge \theta_i} l_i\big(Y^i_{x,t}(s), t - s, \alpha_i(s)\big)\,\mathrm{d}s + u(Y^i_{x,t}(\sigma \wedge \theta_i), t - (\sigma \wedge \theta_i)) \right], \tag{3.13}$$

where $Y^i_{x,t}$ *is the solution of the ode* (3.7) *and the infimum/supremum is taken on all stopping times* θ_i *such that* $Y^i_{x,t}(\theta_i) \in \partial\Omega_i$ *and* $\tau_i \leq \theta_i \leq \bar{\tau}_i$ *where* τ_i *is the first exit time of the trajectory* $Y^i_{x,t}$ *from* Ω_i *and* $\bar{\tau}_i$ *is the one from* $\overline{\Omega}_i$.

The following important result highlights the fundamental alternative: given $x \in \mathcal{H}$, either there exists an optimal strategy consisting in entering in Ω_1 or Ω_2, or all the optimal strategies consist in staying on $\mathcal{H}$ at least for a while.

Theorem 3.8. *Assume* $(\mathbf{H}_\Omega)$ *and* $(\mathbf{H}_\mathrm{C})$*. Let* $v : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ *be a lsc supersolution of* $v_t + H(x,t,Dv) = 0$*. Let* $\phi \in C^1\big(\mathcal{H} \times [0,T]\big)$ *and* (x,t) *be a minimum point of* $(z,s) \mapsto v(z,s) - \phi(z,s)$*. Then, the following alternative holds:*

A) *either there exist* $\eta > 0$*,* $i \in \{1,2\}$ *and a sequence* $x_k \in \overline{\Omega}_i$ *converging to* x *such that* $v(x_k,t) \to v(x,t)$ *and, for each* k*, there exists a control* $\alpha_i^k(\cdot)$ *such that the corresponding trajectory* $Y^i_{x_k,t}(s) \in \overline{\Omega}_i$ *for all* $s \in [0,\eta]$ *and*

$$v(x_k,t) \geq \int_0^\eta l_i\big(Y^i_{x_k,t}(s), t-s, \alpha_i^k(s)\big)\,\mathrm{d}s + v\big(Y^i_{x_k,t}(\eta), t-\eta\big)\ ; \tag{3.14}$$

B) *or there holds*

$$\phi_t(x,t) + H_T\big(x,t,D_\mathcal{H}\phi(x,t)\big) \geq 0. \tag{3.15}$$

Proof. As in [6, Thm. 3.3], we are going to prove that if **A)** does not hold, then necessarily the second possibility holds. Up to a standard modification of ϕ, we may assume that the max is strict. For $\varepsilon > 0$ we consider the function

$$v(z,s) - \tilde{\phi}(z,s) - \delta \mathrm{d}_\mathcal{H}(z) + \frac{\mathrm{d}_\mathcal{H}(z)^2}{\varepsilon^2}\ ,$$

where we recall that $\mathrm{d}_\mathcal{H}(\cdot)$ is the signed distance function to $\mathcal{H}$ as in the proof of Theorem 3.6.

There are two cases: either for ε small enough, the minimum point $(x_\varepsilon, t_\varepsilon)$ lies on $\mathcal{H} \times [0,T]$ and this leads directly to (3.15) as in [6, Thm. 3.3]; or we may assume that for instance, $x_\varepsilon \in \Omega_i$ for ε small enough. In this second case, the argument by contradiction in [6, Thm 3.3. - 2nd case] applies, using Lemma 3.7. □

4 Uniqueness result

We first prove a local comparison result which is based on auxiliary results in the appendix. To this end, we denote by $Q^{(x_0,t_0)}(r,h)$ the open cylinder $Q^{(x_0,t_0)}(r,h) := B(x_0,r) \times (t_0 - h, t_0)$ where $0 < t_0 - h < t_0 < T$, whose parabolic boundary is given by

$$\partial_p Q^{(x_0,t_0)}(r,h) := B(x_0,r) \times \{t_0 - h\} \cup \partial B(x_0,r) \times [t_0 - h, t_0)\,.$$

In the sequel, we assume that $x_0 \in \mathcal{H}$ and that, thanks to $(\mathbf{H}_\Omega)$, r is small enough in order that there exists a $W^{2,\infty}$-diffeomorphism $\Psi = \Psi_{(x_0,r)}$ such that by setting $\tilde{\Omega} := \Psi\big(B(x_0,r))\big)$, we have

$$\Psi\big(\mathcal{H} \cap B(x_0,r)\big) = \{x_N = 0\} \cap \tilde{\Omega}\,.$$

We denote this assumption by $(\mathbf{H}_\Omega^{x_0})$.

Theorem 4.1. *Assume* $(\mathbf{H}_\Omega^{x_0})$ *and* $(\mathbf{H}_\mathrm{C})$*. If* u *and* v *are respectively a bounded usc subsolution and a bounded lsc supersolution of* $w_t + \mathbb{H}^-(x,t,Dw) = 0$ *in* $Q^{(x_0,t_0)}(r,h)$ *. Then*

$$\|(u-v)_+\|_{L^\infty(Q^{(x_0,t_0)}(r,h))} \leq \|(u-v)_+\|_{L^\infty(\partial_p Q^{(x_0,t_0)}(r,h))}. \tag{4.1}$$

Proof. We make the change of variable : $\tilde{u}(x,t) := u(\Psi^{-1}(x),t)$, $\tilde{v}(x,t) := v\big(\Psi^{-1}(x),t\big)$. The functions $\tilde{u},\tilde{v}$ are respectively sub and supersolution of (7.1) with $\tilde{Q} = \tilde{\Omega} \times (t_0 - h, t_0)$, for an Hamiltonian $\tilde{\mathbb{H}}^-$ associated to

$$\tilde{b}_i(x,t,\cdot) := D\Psi(\Psi^{-1}(x))b_i\big(\Psi^{-1}(x),t,\cdot\big)\,,\ \tilde{l}_i(x,t,\cdot) := l_i\big(\Psi^{-1}(x),t,\cdot\big) \quad \text{for } x \in \tilde{\Omega},\ t \in [t_0 - h, t_0]\,.$$

These dynamics and costs satisfy $(\mathbf{H}_\mathrm{C})$ for some new constants denoted by $\tilde{M}_b, \tilde{L}_b, \tilde{M}_l, \tilde{m}_l, \tilde{\delta}$.

We apply Lemma 7.7 which gives (7.7) which is exactly the result we want by making the change back. □

We now turn to one of our main results, which is the

Theorem 4.2. *Assume* $(\mathbf{H}_\Omega)$ *and* $(\mathbf{H}_\mathrm{C})$. *If u is a bounded, usc subsolution of (3.5) and v is a bounded, lsc supersolution of (3.5), satisfying $u(x,0) \leq v(x,0)$ in $\mathbb{R}^N$, then $u \leq v$ in $\mathbb{R}^N \times (0,T)$.*

Proof. We first prove the

Lemma 4.3. *For $K > 0$ large enough, $\psi(x,t) := -Kt-(1+|x|^2)^{1/2}$ satisfies $\psi_t+\mathbb{H}^-(x,t,D\psi) \leq -1$ in $\mathbb{R}^N \times (0,T)$.*

Proof. We just estimate as follows:

$$\psi_t + \mathbb{H}^-(x,t,D\psi) \leq -K + M_b|D\psi| + M_l \leq -K + M_b + M_l\ .$$

Hence taking $K \geq M_b + M_l + 1$ yields the result. □

Using the function ψ of Lemma 4.3, we introduce, for $\mu \in (0,1)$ close to 1, the function $u_\mu(x,t) := \mu u(x,t) + (1-\mu)\psi(x,t)$. Because of the convexity properties of H_1, H_2, H_T, it satisfies $(u_\mu)_t + \mathbb{H}^-(x,t,Du_\mu) \leq -(1-\mu)$. Then we consider

$$M_\mu := \sup_{\mathbb{R}^N \times [0,T]} \big(u_\mu(x,t) - v(x,t)\big)\,.$$

Since $u_\mu(x,t) \to -\infty$ as $|x| \to \infty$ (uniformly with respect to $t \in [0,T]$) and v is bounded, this "sup" is actually a "max" and it is achieved at (x_0,t_0). Notice also that $M_\mu \to M := \sup_{\mathbb{R}^N\times[0,T]} \big(u(x,t) - v(x,t)\big)$ as $\mu \to 1$. We argue by contradiction, assuming that $M > 0$, which implies that $M_\mu > 0$ for μ close enough to 1. From now on, we assume that we have chosen such a μ and therefore $M_\mu > 0$.

Next we remark that $t_0 > 0$ since $u_\mu(x,0) - v(x,0) \leq 0$ in $\mathbb{R}^N$ and we first treat the case when $x_0 \in \mathcal{H}$. In that way, since $(\mathbf{H}_\Omega)$ holds, we can choose $r > 0$, small enough in order that $(\mathbf{H}_\Omega^{x_0})$ holds. On the other hand, we choose any h such that $t_0 - h \geq 0$, say $h = t_0$.

The next step consists in introducing the function

$$\bar{u}_\mu(x,t) := u_\mu(x,t) + (1-\mu)^2\big(t - t_0 - |x-x_0|^2\big)\ .$$

We claim that $\bar{u}_\mu$ is a subsolution of $(\bar{u}_\mu)_t + \mathbb{H}^-(x,t,D\bar{u}_\mu) = 0$ for μ close enough to 1. Indeed, a direct computation gives

$$\begin{aligned}(\bar{u}_\mu)_t + \mathbb{H}^-(x,\bar{u}_\mu,D\bar{u}_\mu) &\le (u_\mu)_t + \mathbb{H}^-(x,u_\mu,Du_\mu) + (1-\mu)^2\{1+2M_b r\} \\ &\le -(1-\mu) + (1-\mu)^2\{1+2M_b r\} \le 0\end{aligned}$$

for μ sufficiently close to 1.

Thus, we use Theorem 4.1 with the pair of sub/supersolution $(\bar{u}_\mu, v)$ and we obtain in particular

$$M_\mu = u_\mu(x_0,t_0) - v(x_0,t_0) = \bar{u}_\mu(x_0,t_0) - v(x_0,t_0) \le \|(\bar{u}_\mu - v)_+\|_{L^\infty(\partial_p Q^{(x_0,t_0)}(r,h))} .$$

However, on the parabolic boundary $(\bar{u}_\mu - v) < M_\mu$. Indeed, on $\partial B(x,r)\times(t_0-h,t_0)$, we have

$$\bar{u}_\mu(x,t) - v(x,t) = u_\mu(x,t) - v(x,t) + (1-\mu)^2\big(t-t_0-r^2\big) \le M_\mu - (1-\mu)^2 r^2 ,$$

while on $B(x_0,r)\times\{t_0-h\}$,

$$\bar{u}_\mu(x,t) - v(x,t) = u_\mu(x,t) - v(x,t) + (1-\mu)^2\big(t-t_0-|x-x_0|^2\big) \le M_\mu - (1-\mu)^2 h .$$

This gives a contradiction.

We can argue in the same way if $x_0 \in \Omega_1$ or $x_0 \in \Omega_2$: in fact this is even easier since we may choose r such that either $\overline{B}(x_0,r)\subset\Omega_1$ or $\overline{B}(x_0,r)\subset\Omega_2$; with this choice we only deal with classical Hamilton-Jacobi Equations without discontinuities and we have just to apply classical results.

The contradiction shows that $M \le 0$ and the proof is complete. □

As a consequence, we have the following

Theorem 4.4. *Assume* $(\mathbf{H}_g)$, $(\mathbf{H}_\Omega)$ *and* $(\mathbf{H}_C)$. *Then*
(i) The value function $\mathbf{U}^-$ *is continuous and the unique solution of*

$$u_t + \mathbb{H}^-(x,t,Du) = 0 \text{ in } \mathbb{R}^N\times(0,T) , \tag{4.2}$$

$$u(x,0) = g(x) \text{ in } \mathbb{R}^N . \tag{4.3}$$

(ii) $\mathbf{U}^-$ *is the minimal supersolution of* (1.3)-(1.5)-(1.6)-(4.3). *The value function* $\mathbf{U}^+$ *is also continuous and the maximal subsolution of* (1.3)-(1.5)-(1.6)-(4.3).

Proof. The proof of (i) is a direct consequence of Theorem 3.3 and 4.2 : indeed $(\mathbf{U}^-)^*$ and $(\mathbf{U}^-)_*$ are respectively sub and supersolution of (4.2) by Theorem 3.3 and since $(\mathbf{U}^-)^*(x,0) = (\mathbf{U}^-)_*(x,0) = g(x)$ in $\mathbb{R}^N$, Theorem 4.2 implies that $(\mathbf{U}^-)^* \le (\mathbf{U}^-)_*$ in $\mathbb{R}^N\times[0,T]$, which implies that $\mathbf{U}^-$ is continuous because $(\mathbf{U}^-)_* \le \mathbf{U}^- \le (\mathbf{U}^-)^*$ in $\mathbb{R}^N\times[0,T]$ and therefore $(\mathbf{U}^-)_* = \mathbf{U}^- = (\mathbf{U}^-)^*$ in $\mathbb{R}^N\times[0,T]$. As a consequence $\mathbf{U}^-$ being both upper and lower semicontinuous, it is continuous. The uniqueness is a direct consequence of Theorem 4.2.

For (ii), the first part is also a direct consequence of Theorem 4.2 since any supersolution of (1.3)-(1.5)-(1.6)-(4.3) is a supersolution of (4.2)-(4.3).

Finally, for $\mathbf{U}^+$, we follow the same idea as for $\mathbf{U}^-$ above and of [6] : if u is a subsolution of (1.3)-(1.5)-(1.6)-(4.3), then by Theorem 3.6, it satisfies

$$u_t + H_T^{\mathrm{reg}}(x,t,Du) \leq 0 \quad \text{on } \mathcal{H} ,$$

and in order to compare it with the supersolution $(\mathbf{U}^+)_*$, we use Theorem 3.5 (instead of Theorem 3.8 for the supersolutions in the case of $\mathbb{H}^-$) together with the regularization of the appendix (done on $\mathbb{H}^+$ and not $\mathbb{H}^-$). We skip the details since it is a straightforward adaptation of the proof of Theorems 4.1-4.2.

Notice that, as a consequence, we have $(\mathbf{U}^+)^* \leq (\mathbf{U}^+)_*$ in $\mathbb{R}^N \times [0,T]$ since $(\mathbf{U}^+)^*$ is a subsolution of (1.3)-(1.5)-(1.6)-(4.3), which implies the continuity of $\mathbf{U}^+$. □

Remark 4.5. *We emphasize the key role of Theorem 3.5:* $\mathbf{U}^+$ *is the only supersolution of the* $\mathbb{H}^+$*-equation for which we have such a property and this is why we do not have a complete comparison result for this equation (contrary to the* $\mathbb{H}^-$ *one).*

5 Stability

In this section we prove stability results when we have a sequence of dynamics and costs $b_i^\varepsilon, l_i^\varepsilon, g^\varepsilon$ converging locally uniformly. Let us begin with a standard stability result for sub/super solutions.

Theorem 5.1. *Assume* $(\mathbf{H}_\Omega)$ *and that, for all* $\varepsilon > 0$, $b_1^\varepsilon, b_2^\varepsilon, l_1^\varepsilon, l_2^\varepsilon$ *satisfy* $(\mathbf{H}_\mathrm{C}^1)$-$(\mathbf{H}_\mathrm{C}^3)$ *with constants uniforms in* ε*. Let* H_i^ε $(i = 1,2)$ *and* H_T^ε *be defined as in* (1.4) *and* (3.1) *respectively with these dynamics and costs. If*

$$(b_1^\varepsilon, b_2^\varepsilon, l_1^\varepsilon, l_2^\varepsilon) \to (b_1, b_2, l_1, l_2) \text{ locally uniformly in } \mathbb{R}^N \times [0,T] \times A\,,$$
$$g^\varepsilon \to g \text{ locally uniformly in } \mathbb{R}^N\,,$$

then the following holds

(i) if, for all $\varepsilon > 0$, v_ε *is a lsc supersolution of*

$$u_t + \mathbb{H}_\varepsilon^-(x,t,Du) = 0 \text{ in } \mathbb{R}^N \times (0,T), \tag{5.1}$$

then $\underline{v} = \liminf_* v_\varepsilon$ *is a lsc supersolution of*

$$u_t + \mathbb{H}^-(x,t,Du) = 0 \text{ in } \mathbb{R}^N \times (0,T), \tag{5.2}$$

where $\mathbb{H}^-$ *is defined as in* (1.4) *and* (3.1) *through the functions* (b_1, b_2) *and* (l_1, l_2)*.*

(ii) If, for $\varepsilon > 0$, u_ε *is an usc subsolution of* (5.1) *and if* b_1, b_2 *satisfy* $(\mathbf{H}_\mathrm{C}^4)$ *then* $\bar{u} = \limsup^* u_\varepsilon$ *is a subsolution of* (5.2)*.*

We point out the unusual form of this stability result : if for supersolutions, the half-relaxed limit result holds true, it is not the case anymore in general for the subsolution. This is related to the H_T inequality which sees only the subsolutions on $\mathcal{H}$. For exemple, if $\mathcal{H} = \{x \in \mathbb{R}^N : x_N = 0\}$ and if $u_\varepsilon(x) = \sin(x_N/\varepsilon)$, then $\limsup^* u_\varepsilon(x,0) \equiv 1$ on $\mathcal{H}$ while $u_\varepsilon(x,0) \equiv 0$. In this example it is clear that the $\limsup^* u_\varepsilon$ comes from the value of u_ε outside $\mathcal{H}$ and it is clear that one cannot recover an H_T-inequality which sees only the values on $\mathcal{H}$. Assumption $(\mathbf{H}_\mathrm{C}^4)$ prevents these pathological situations to hold.

Proof. This proof follows almost completely from standard arguments for stability results on viscosity solutions (see, for instance [5]): we apply the standard stability results in $\mathbb{R}^N$ for the Hamiltonian defined in the introduction, and in $\mathcal{H}$ for H_T. Since we can flatten the boundary this last result is essentially a result in $\mathbb{R}^{N-1}$.

The only case that need to be detailed is the proof of (ii) and more precisely $\bar{u}$ fulfilling the inequality $u_t + H_T(x,t,Du) \leq 0$ on $\mathcal{H}$. To do so, we use the

Lemma 5.2. *Under the assumptions of Theorem 5.1 (ii), H_T^ε converges to H_T locally uniformly.*

We postpone the proof and return to the proof of Theorem 5.1 (ii). We first remark that, thanks to $(\mathbf{H}_\Omega)$, we can argue as in the proof of uniqueness and suppose that we are working with $\mathcal{H} = \{x_N = 0\}$ (see assumption $(\mathbf{H}_\Omega^{x_0})$ and its consequences).

If $\phi \in C^1(\mathcal{H} \times [0,T])$ and if (x_0', t_0) is a strict local maximum point of $\bar{u}(y', 0, s) - \phi(y', s)$ in $\mathcal{H} \times [0,T]$, our aim is to prove that

$$\phi_t(x_0', t_0) + H_T\big((x_0', 0), t_0, D_{\mathcal{H}}\phi(x_0', t_0)\big) \leq 0\ . \tag{5.3}$$

By the definition of $\limsup^* u_\varepsilon$, there exists a sequence $(\bar{x}_\varepsilon, \bar{t}_\varepsilon)$ converging to $(x_0', 0, t_0)$ such that $\bar{u}(x_0', 0, t_0) = \lim_\varepsilon u_\varepsilon(\bar{x}_\varepsilon, \bar{t}_\varepsilon)$. If $(\bar{x}_\varepsilon)_N \neq 0$, we set $K_\varepsilon = |(\bar{x}_\varepsilon)_N|^{-1/2}$, otherwise $K_\varepsilon = \varepsilon^{-1}$. Notice that $K_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

We consider the function $\psi_\varepsilon(x,t) := u_\varepsilon(x,s) - \phi(x', s) - K_\varepsilon |x_N|$. By classical techniques, using that $\psi_\varepsilon(\bar{x}_\varepsilon, \bar{t}_\varepsilon) \to \bar{u}(x', 0, t_0) - \phi(x', t_0)$ (this key property justifies the choice of K_ε), one proves easily that there exists a sequence $(x_\varepsilon, t_\varepsilon)$ of maximum points of ψ_ε which converges to $(x_0', 0, t_0)$.

If $x_\varepsilon \in \Omega_1 \subset \{x \in \mathbb{R}^N : \ x_N > 0\}$, $x \mapsto |x_N|$ is smooth in a neighborhood of x_ε and, since u_ε is an usc subsolution of (5.1), we have

$$\phi_t(x_\varepsilon', t_\varepsilon) + H_1^\varepsilon(x_\varepsilon, t_\varepsilon, D_{\mathcal{H}}\phi(x_\varepsilon', t_\varepsilon) + K_\varepsilon \mathbf{e}_N) \leq 0$$

but, recalling that $K_\varepsilon \to +\infty$ as $\varepsilon \to 0$, this inequality cannot hold for ε small enough because of $(\mathbf{H}_\mathrm{C}^4)$. To be more precise, since the b_i^ε converge locally uniformly to b_i which statisfy $(\mathbf{H}_\mathrm{C}^4)$, we can take a uniform $\delta = \tilde{\delta}$ in Lemma 7.1 which proves the claim.

In the same way x_ε cannot be in Ω_2. As a consequence, x_ε is on $\mathcal{H}$ and is a maximum point of $(y', s) \mapsto u_\varepsilon(y', 0, s) - \phi(y', s)$. But u_ε is an usc subsolution of (5.1), therefore the H_T^ε-inequality holds and we conclude in the classical way using Lemma 5.2. ☐

Proof of Lemma 5.2. By the definition of H_T^ε,

$$H_T^\varepsilon(x,t,p) := \sup_{A_0(x,t)} \big\{ -\big\langle b_{\mathcal{H}}^\varepsilon(x,t,a), p\big\rangle - l_{\mathcal{H}}^\varepsilon(x,t,a)\big\}.$$

If $x \in \mathcal{H}$, $t \in (0,T)$ and if $(x_\varepsilon, t_\varepsilon)_\varepsilon$ is a sequence in $\mathcal{H} \times (0,T)$ converging to (x,t) and if $p_\varepsilon \to p$, we use this definition to write

$$H_T^\varepsilon(x_\varepsilon, t_\varepsilon, p_\varepsilon) = -\big\langle b_{\mathcal{H}}^\varepsilon(x_\varepsilon, t_\varepsilon, a_\varepsilon), p_\varepsilon\big\rangle - l_{\mathcal{H}}^\varepsilon(x_\varepsilon, t_\varepsilon, a_\varepsilon) \geq -\big\langle b_{\mathcal{H}}^\varepsilon(x_\varepsilon, t_\varepsilon, a), p_\varepsilon\big\rangle - l_{\mathcal{H}}^\varepsilon(x_\varepsilon, t_\varepsilon, a) \tag{5.4}$$

for any $a \in A_0(x_\varepsilon, t_\varepsilon)$.

Again by definition, we have

$$b^\varepsilon_{\mathcal{H}}(x_\varepsilon, t_\varepsilon, a_\varepsilon) = \mu_\varepsilon b_1(x_\varepsilon, t_\varepsilon, \alpha^\varepsilon_1) + (1-\mu_\varepsilon) b_2(x_\varepsilon, t_\varepsilon, \alpha^\varepsilon_2)\,,$$

and extracting subsequences, we can assume that $b^\varepsilon_{\mathcal{H}}(x_\varepsilon, t_\varepsilon, a_\varepsilon)$ converges to $b_{\mathcal{H}}(x,t,\bar{a})$. In the same way, $l^\varepsilon_{\mathcal{H}}(x_\varepsilon, t_\varepsilon, a) \to l_{\mathcal{H}}(x,t,\bar{a})$. It remains to show that

$$H_T(x,t,p) = -\big\langle b_{\mathcal{H}}(x,t,\bar{a}), p\big\rangle - l_{\mathcal{H}}(x,t,\bar{a})\;.$$

This can be done using Inequality (5.4) and the arguments of Lemma 7.2 : if

$$H_T(x,t,p) = -\big\langle b_{\mathcal{H}}(x,t,\hat{a}), p\big\rangle - l_{\mathcal{H}}(x,t,\hat{a})\;,$$

we can build a sequence $\tilde{a}_\varepsilon \in A_0(x_\varepsilon, t_\varepsilon)$ such that

$$-\big\langle b^\varepsilon_{\mathcal{H}}(x_\varepsilon, t_\varepsilon, \tilde{a}_\varepsilon), p_\varepsilon\big\rangle - l^\varepsilon_{\mathcal{H}}(x_\varepsilon, t_\varepsilon, \tilde{a}_\varepsilon) \to -\big\langle b_{\mathcal{H}}(x,t,\hat{a}), p\big\rangle - l_{\mathcal{H}}(x,t,\hat{a})\;.$$

Passing to the limit in the inequality (5.4) with $a = \tilde{a}_\varepsilon$, we have the desired conclusion. $\square$

We now turn to the stability of the minimal and maximal solutions. To do so, we denote by $\mathcal{T}^\varepsilon_{x,t}$ [resp. $\mathcal{T}^{\mathrm{reg},\varepsilon}_{x,t}$] the set of admissible [resp. admissible and regular] trajectories associated to the dynamics b^ε_i , $i=1,2$. We also define the costs functionals J^ε as in (2.5), but with ℓ^ε and g^ε.

Lemma 5.3. *Under the assumptions of Theorem 5.1, if for any $\varepsilon > 0$, $(X^\varepsilon, a^\varepsilon) \in \mathcal{T}^\varepsilon_{x,t}$, the following holds*

i) There exists a subsequence $(X^{\varepsilon_n}, a^{\varepsilon_n})_n$ converging to an admissible trajectory $(X,a) \in \mathcal{T}_{x,t}$. More precisely, $X^{\varepsilon_n} \to X$ uniformly in $[0,T]$ and

$$J(x,t;(X^{\varepsilon_n}, a^{\varepsilon_n})) \to J(x,t,(X,a)) \quad \textit{uniformly in } [0,T]\,.$$

ii) If, moreover, $(X^\varepsilon, a^\varepsilon) \in \mathcal{T}^{\mathrm{reg},\varepsilon}_{x,t}$ for any $\varepsilon > 0$ (i.e., the trajectories are regular), then we have a subsequence for which the limit trajectory is also regular: $(X,a) \in \mathcal{T}^{\mathrm{reg}}_{x,t}$.

iii) The results in i) (and ii)) hold true also if we assume that for each $\varepsilon > 0$, the trajectories $(X^\varepsilon, a^\varepsilon) \in \mathcal{T}_{x_\varepsilon,t_\varepsilon} (\in \mathcal{T}^{\mathrm{reg}}_{x_\varepsilon,t_\varepsilon})$, and we assume that $(x_\varepsilon, t_\varepsilon) \to (x,t)$ as $\varepsilon \to 0$.

Proof. The proof of *i)* is almost standard and we only provide it for the reader's convenience. On the contrary, the proof of *ii)* reveals unexpected difficulties (but which come from the particular features of the control problem).

PROOF OF *i)* — Since we want to pass to the limit both on the dynamic and the cost, we rewrite the differential inclusion in a different way, taking into account both at the same time.

We fix (x,t). Since the trajectories go backward in time, we introduce the variable $\sigma(s) := t - s$, starting at $\sigma(0) = t$. Then, for any $\varepsilon > 0$, using the admissible trajectory $(X^\varepsilon, a^\varepsilon)$ we set

$$Y^\varepsilon(s) := \int_0^s \ell^\varepsilon\big(X^\varepsilon(\tau), \sigma(\tau), a^\varepsilon(\tau)\big)\,\mathrm{d}\tau$$

where the Lagrangian ℓ^ε is defined as in (2.6), but with $l_1^\varepsilon, l_2^\varepsilon$. In order to take into acount both X^ε and Y^ε at the same time and the function $\sigma(\cdot)$, we consider the mixed variable $Z := (X, Y, \sigma) \in \mathbb{R}^N \times \mathbb{R} \times [0,T]$, and translate the differential inclusion in terms of Z.

To do so, we use $(\mathbf{H}_{\mathrm{C}}^3)$ and introduce, for $i = 1, 2$, the sets

$$\mathbf{BL}_i^\varepsilon(Z) := \left\{ \big(b_i^\varepsilon(X, \sigma, \alpha_i), l_i^\varepsilon(X, \sigma, \alpha_i), -1\big) : \alpha_i \in A_i \right\} ,$$

$$\mathcal{BL}^\varepsilon(Z) := \begin{cases} \mathbf{BL}_i^\varepsilon(Z) & \text{if } X \in \Omega_i \,, \\ \overline{\mathrm{co}}\big(\mathbf{BL}_1^\varepsilon(Z) \cup \mathbf{BL}_2^\varepsilon(Z)\big) & \text{if } X \in \mathcal{H} \,. \end{cases}$$

It turns out that the triple $Z^\varepsilon := (X^\varepsilon, Y^\varepsilon, \sigma)$ is a solution of the differential inclusion

$$\dot{Z}^\varepsilon(s) \in \mathcal{BL}^\varepsilon\big(Z^\varepsilon(s)\big) \quad \text{for a.e. } s \in [0,t) \,, \quad \text{with } Z^\varepsilon(0) = (x, 0, t) \,.$$

We first notice that since the $b_i^\varepsilon, l_i^\varepsilon$ are uniformly bounded, the Z^ε are equi-Lipschitz and equi-bounded on $[0,T]$. Therefore we can extract a subsequence (denoted by Z^{ε_n}) which converges uniformly on $[0,T]$ to some $Z = (X, Y, \sigma)$. Moreover, for any given $\delta > 0$ and for $\varepsilon > 0$ small enough, we have, for any $s \in (0,t)$

$$\mathcal{BL}^{\varepsilon_n}(Z^{\varepsilon_n}) \subset \mathcal{BL}(Z) + \delta B_{N+2} \,,$$

where B_{N+2} is the unit ball in $\mathbb{R}^{N+2}$, centered at the origin. Using this information, it is immediate that $\dot{Z}(s) \in \mathcal{BL}\big(Z(s)\big)$ almost everywhere. In particular the limit trajectory is admissible: there exists a control $a(\cdot)$ such that $\big(X, a\big) \in \mathcal{T}_{x,t}$. (See Filippov's Lemma [1, Theorem 8.2.1] or the proof of Theorem 2.1 in [6]).

We deduce also that necessarily,

$$Y^{\varepsilon_n}(s) \to Y(s) = \int_0^s \ell\big(X(\tau), \sigma(\tau), a(\tau)\big)\, \mathrm{d}\tau \quad \text{uniformly in } [0,t] \,.$$

Finally, since $g^\varepsilon \to g$ locally uniformly in $\mathbb{R}^N$ and $X^{\varepsilon_n} \to X$ uniformly on $[0,T]$, we deduce that $J(x, t; (X^{\varepsilon_n}, a^{\varepsilon_n}))$ converges to $J(x, t, (X, a))$ uniformly with respect to $t \in [0,T]$.

Proof of $ii)$ — The difficulty comes from two facts: the first one is that we have to deal with weak convergences in the $b_i^\varepsilon, b_{\mathcal{H}}^\varepsilon$-terms but the problem is increased by the fact that some pieces of the trajectory $X(\cdot)$ on $\mathcal{H}$ can be obtained as limits of trajectories $X^\varepsilon(\cdot)$ which lie either on $\mathcal{H}$, Ω_1 or Ω_2. In other words, the indicator functions $\mathbb{1}_{\{X^\varepsilon \in \mathcal{H}\}}(\cdot)$ do not converge to $\mathbb{1}_{\{X \in \mathcal{H}\}}(\cdot)$, and similarly the $\mathbb{1}_{\{X^\varepsilon \in \Omega_i\}}(\cdot)$ do not converge to $\mathbb{1}_{\{X \in \Omega_i\}}(\cdot)$. We proceed in three steps.

Step 1. We first recall that

$$\dot{X}^\varepsilon(s) = \sum_{i=1,2} b_i^\varepsilon\big(X^\varepsilon(s), \sigma(s), \alpha_i^\varepsilon(s)\big) \mathbb{1}_{\{X^\varepsilon \in \Omega_i\}}(s) + b_{\mathcal{H}}^\varepsilon\big(X^\varepsilon(s), \sigma(s), a^\varepsilon(s)\big) \mathbb{1}_{\{X^\varepsilon \in \mathcal{H}\}}(s)$$

converges weakly ($i.e.$ in $L^\infty(0,T)$ weak$-*$) to

$$\dot{X}(s) = \sum_{i=1,2} b_i\big(X(s), \sigma(s), \alpha_i(s)\big) \mathbb{1}_{\{X \in \Omega_i\}}(s) + b_{\mathcal{H}}\big(X(s), \sigma(s), a(s)\big) \mathbb{1}_{\{X \in \mathcal{H}\}}(s) \,, \tag{5.5}$$

for some control $a(\cdot)$ such that $(X,a)\in\mathcal{T}_{x,t}$. This weak convergence does not create any difficulty if $X(s)$ is in Ω_i for $i=1,2$ but it is a little bit more complicated if $X(s)\in\mathcal{H}$ since the term $b_{\mathcal{H}}\big(X(s),\sigma(s),a(s)\big)\mathbb{1}_{\{X\in\mathcal{H}\}}(s)$ is a weak limit of

$$\sum_{i=1,2} b_i^\varepsilon\big(X^\varepsilon(s),\sigma(s),\alpha_i^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon\in\Omega_i\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s)+b_{\mathcal{H}}^\varepsilon\big(X^\varepsilon(s),\sigma(s),a^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon\in\mathcal{H}\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s)\ ,$$

and we have to check that both terms cannot generate singular strategies. In order to examine carefully the mechanism of the weak convergence on $\mathcal{H}$, we write, for $0\le\tau\le t$

$$X^\varepsilon(\tau)-x=\sum_{i=1,2}\int_0^\tau b_i^\varepsilon\big(X^\varepsilon(s),\sigma(s),\alpha_i^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon\in\Omega_i\}}(s)\,\mathrm{d}s+\int_0^\tau b_{\mathcal{H}}^\varepsilon\big(X^\varepsilon(s),\sigma(s),a^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon\in\mathcal{H}\}}(s)\,\mathrm{d}s\,,$$

and we use a slight modification of the procedure leading to relaxed control as follows. We write

$$\int_0^\tau b_1^\varepsilon\big(X^\varepsilon(s),\sigma(s),\alpha_1^\varepsilon(s)\big)\mathbb{1}_{\{X^\varepsilon\in\Omega_1\}}(s)\,\mathrm{d}s=\int_0^\tau\int_{A_1} b_1^\varepsilon\big(X^\varepsilon(s),\sigma(s),\alpha_1\big)\,\nu_1^\varepsilon(s,\,\mathrm{d}\alpha_1)\,\mathrm{d}s\,,$$

where $\nu_1^\varepsilon(s,\cdot)$ stands for the measure defined on A_1 by $\nu_1^\varepsilon(s,E)=\delta_{\alpha_1^\varepsilon}(E)\mathbb{1}_{\{X^\varepsilon\in\Omega_1\}}(s)$, for any Borelian set $E\subset A_1$. Similarly we define ν_2^ε and $\nu_{\mathcal{H}}^\varepsilon$ for the other terms. Notice that $\nu_{\mathcal{H}}^\varepsilon$ is a bit more complex measure since it concerns controls of the form $a=(\alpha_1,\alpha_2,\mu)$ on A, but it works as for ν_1^ε so we omit the details.

Note that, for any s, $\nu_1^\varepsilon(s,A_1)+\nu_2^\varepsilon(s,A_2)+\nu_{\mathcal{H}}^\varepsilon(s,A)=1$ and therefore the measures $\nu_1^\varepsilon(s,\cdot),\nu_2^\varepsilon(s,\cdot),\nu_{\mathcal{H}}^\varepsilon(s,\cdot)$ are uniformly bounded in ε. Up to successive extractions of subsequences, they all converge weakly to some measures ν_1, ν_2, $\nu_{\mathcal{H}}$. Since the total mass is 1, we obtain in the limit $\nu_1(s,A_1)+\nu_2(s,A_2)+\nu_{\mathcal{H}}(s,A)=1$. Using that (also up to extraction form the proof of $i)$ above), X^ε converges uniformly on $[0,t]$ and the local uniform convergence of the b_i^ε, we get that

$$\int_{A_1} b_1^\varepsilon\big(X^\varepsilon(s),\sigma(s),\alpha_1\big)\,\nu_1^\varepsilon(s,\,\mathrm{d}\alpha_1)\ \underset{\varepsilon\to0}{\longrightarrow}\ \int_{A_1} b_1\big(X(s),\sigma(s),\alpha_1\big)\,\nu_1(s,\,\mathrm{d}\alpha_1),\ \text{weakly in } L^\infty(0,T)\,.$$

Introducing $\pi_1(s):=\int_{A_1}\nu_1(s,\,\mathrm{d}\alpha_1)$ and using the convexity of A_1 to gether with measurable selection argument (see [1, Theorem 8.1.3]), the last integral can be written as $b_1\big(X(s),\sigma(s),\alpha_1^\sharp(s)\big)\pi_1(s)$ for some control $\alpha_1^\sharp\in L^\infty(0,T;A_1)$. The same procedure for the other two terms provides the controls $\alpha_2^\sharp(\cdot)$, $a^\sharp(\cdot)$ and functions $\pi_2(\cdot)$, $\pi_{\mathcal{H}}(\cdot)$. In principle, those controls can be different from $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $a(\cdot)$ but this will not be a problem since $\alpha_1^\sharp(\cdot)$, $\alpha_2^\sharp(\cdot)$, $a^\sharp(\cdot)$ are just intermediate controls which are used to prove that the strategy $a(\cdot)$ is regular.

Step 2. We then deal with the b_i-terms. If $d_{\Omega_i}(x)$ denotes the distance from x to Ω_i then $d_{\Omega_i}(X^\varepsilon)$ is a sequence of Lipschitz continuous functions which converges uniformly to $d_{\Omega_i}(X)$ and, up to an additional extraction of subsequence, we may assume that the derivatives converges weakly in L^∞ (weak–$*$ convergence). As a consequence, $\frac{d}{ds}\big[d_{\Omega_i}(X^\varepsilon)\big]\mathbb{1}_{\{X\in\mathcal{H}\}}$ converges weakly to $\frac{d}{ds}\big[d_{\Omega_i}(X)\big]\mathbb{1}_{\{X\in\mathcal{H}\}}$.

In order to use this convergence we have to compute $\frac{d}{ds}\big[d_{\Omega_i}(X^\varepsilon)\big]$. Using the extension of $\mathbf{n}_i$ outside $\mathcal{H}$ in such a way that $Dd_{\Omega_i}(x)=-\mathbf{n}_i(x)\mathbb{1}_{\{x\in\Omega_j\}}$, together with the regularity of Ω_i and Stampacchia's Theorem we have

$$\frac{d}{ds}\big[d_{\Omega_i}(X^\varepsilon)\big]=\dot X^\varepsilon(s)\cdot\mathbf{n}_i(X^\varepsilon(s))\mathbb{1}_{\{X^\varepsilon\in\Omega_j\}}(s)\quad\text{ for almost all } s\in(0,T).$$

Indeed, on one hand, the distance function is regular outside $\mathcal{H}$ while, on the other hand, $\dot{X}^\varepsilon(s)\cdot\mathbf{n}_i(X^\varepsilon(s))=0$ a.e. on $\mathcal{H}$. Therefore the above convergence reads, for $i\neq j$,

$$\dot{X}^\varepsilon(s)\cdot\mathbf{n}_i(X^\varepsilon(s))\mathbb{1}_{\{X^\varepsilon\in\Omega_j\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s)\longrightarrow\dot{X}(s)\cdot\mathbf{n}_i(X(s))\mathbb{1}_{\{X\in\Omega_j\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s)=0$$

in $L^\infty(0,T)$ weak–$*$, or equivalently using the above expression of $\dot{X}^\varepsilon(s)$,

$$b_j^\varepsilon\big(X^\varepsilon(s),\sigma(s),\alpha_j^\varepsilon(s)\big)\cdot\mathbf{n}_j(X^\varepsilon(s))\mathbb{1}_{\{X^\varepsilon\in\Omega_j\}}(s)\mathbb{1}_{\{X\in\mathcal{H}\}}(s)\longrightarrow 0\quad\text{in } L^\infty(0,T)\text{ weak}-*\ .$$

This implies that for $i=1,2$

$$b_i\big(X(s),\sigma(s),\alpha_i^\sharp(s)\big)\cdot\mathbf{n}_i(X(s))\,\pi_i(s)=0\ \text{ a.e. on } \{X(s)\in\mathcal{H}\}\ ,\tag{5.6}$$

which means that, in these terms, the involved dynamics are regular since they are tangential (provided we take the $\alpha_i^\sharp$ as controls).

Step 3. We are now ready to prove that $(X,a)\in\mathcal{T}_{x,t}^{\rm reg}$, *i.e.* the dynamic in the $b_\mathcal{H}$-term of (5.5) is regular. To do so, we introduce the convex set of regular dynamics for $z\in\mathcal{H}$ and $0\le s\le t$ that we denote by

$$K(z,s):=\big\{b_\mathcal{H}\big(z,s,a_*\big)\,,a_*\in A_0^{\rm reg}(z,s)\big\}\subset\mathbb{R}^N\,.$$

We notice that, for any $z\in\mathcal{H}$ and $s\in[0,T]$, $K(z,s)$ is closed and convex, and the mapping $(z,s)\mapsto K(z,s)$ is continuous on $\mathcal{H}$ for the Hausdorff distance. Then, for any $\eta>0$, we consider the subset of $[0,t]$ consisting of all times for which one has singular (η-enough) dynamics for the control $a(\cdot)$, namely

$$E_{\rm sing}^\eta:=\Big\{s\in[0,t]:X(s)\in\mathcal{H}\text{ and }\operatorname{dist}\Big(b_\mathcal{H}\big(X(s),t-s,a(s)\big);K\big(X(s),t-s\big)\Big)\ge\eta\Big\}$$

and we argue by contradiction, assuming that, for some $\eta>0$, $|E_{\rm sing}^\eta|>0$.

If we take $s\in E_{\rm sing}^\eta$, since $K(X(s),t-s)$ is closed and convex, there exists an hyperplane separating $b_\mathcal{H}\big(X(s),t-s,a(s)\big)$ from $K(X(s),t-s)$ and we may construct an affine function $\Psi_s:\mathbb{R}^N\to\mathbb{R}$ of the form $\Psi_s(z)=c(s)\cdot z+d(s)$ such that

$$\Psi_s\Big(b_\mathcal{H}\big(X(s),t-s,a(s)\big)\Big)\le-1\text{ if }s\in E_{\rm sing}^\eta\,,\quad\Psi_s\ge+1\text{ on }K\big(X(s),t-s\big)\,.$$

Since the mapping $s\mapsto b_\mathcal{H}\big(X(s),t-s,a(s)\big)$ is measurable and $s\mapsto K\big(X(s),t-s\big)$ is continuous (this can be seen as a consequence of Remark 7.5), we can assume that the coefficients $c(s),d(s)$ are in L^∞ (they are bounded because the distance $\eta>0$ is fixed). Hence we may consider the integral

$$I^\varepsilon:=\int_0^t\big(\Psi_s(\dot{X}^\varepsilon(s)\big)\mathbb{1}_{E_{\rm sing}^\eta}(s)\,\mathrm{d}s\,.$$

On the one hand, since Ψ_s is an affine function, by weak convergence of $\dot{X}^\varepsilon$ as $\varepsilon\to0$ and the fact that $\dot{X}=b_\mathcal{H}$ when $s\in E_{\rm sing}^\eta$, we have

$$I^\varepsilon\to\int_0^t\Psi_s(\dot{X}(s))\mathbb{1}_{E_{\rm sing}^\eta}(s)\,\mathrm{d}s=\int_0^t\Psi_s\Big(b_\mathcal{H}\big(X(s),t-s,a(s)\big)\Big)\mathbb{1}_{E_{\rm sing}^\eta}(s)\,\mathrm{d}s\le-|E_{\rm sing}^\eta|<0\,.$$

On the other hand, we can also use the decomposition

$$
\begin{aligned}
I^\varepsilon = &\int_0^t c(s)\mathbb{1}_{E^\eta_{\text{sing}}}(s)\left(\sum_{i=1,2} b_i^\varepsilon\big(X^\varepsilon(s), t-s, \alpha_i^\varepsilon\big)\mathbb{1}_{\{X^\varepsilon\in\Omega_i\}}(s)\right)\mathrm{d}s \\
&+\int_0^t c(s)\mathbb{1}_{E^\eta_{\text{sing}}}(s) b^\varepsilon_{\mathcal{H}}(X^\varepsilon(s), t-s, a^\varepsilon(s))\mathbb{1}_{\{X^\varepsilon\in\mathcal{H}\}}(s)\,\mathrm{d}s + \int_0^t d(s)\mathbb{1}_{E^\eta_{\text{sing}}}(s)\,\mathrm{d}s\,.
\end{aligned}
\tag{5.7}
$$

Notice that, in the second term above, $a^\varepsilon(\cdot)$ is a regular control for the trajectory X^ε, and we want to keep this property in the limit as $\varepsilon\to 0$. To do so the key remark is the following: fix $\varepsilon>0$ and $s\in[0,t]$ for each $a^\varepsilon(s)\in A_0^{\text{reg}}(X^\varepsilon(s),t-s)$ there exists a $\tilde a^\varepsilon(s)\in A_0^{\text{reg}}(X(s),t-s)$ such that

$$
b^\varepsilon_{\mathcal{H}}(X^\varepsilon(s), t-s, a^\varepsilon(s)) - b^\varepsilon_{\mathcal{H}}(X(s), t-s, \tilde a^\varepsilon(s)) = o_\varepsilon(1),
$$

where $o_\varepsilon(1)$ represents any quantity which goes to zero as $\varepsilon\to 0$. Indeed, for $\varepsilon>0$, we can apply Remark 7.5 for each s fixed and a measurable selection argument (see Filippov's Lemma [1, Theorem 8.2.10]) to obtain the existence of the control $a^\varepsilon(s)\in A_0^{\text{reg}}(X^\varepsilon(s),t-s)$ and then deduce the estimate by recalling that X^ε converges uniformly to X. Moreover, by construction and using again a measurable selection argument (see Filippov's Lemma [1, Theorem 8.2.10]), there exists a control $a_\star(s)\in K(X(s),t-s)$ such that

$$
c(s)b_{\mathcal{H}}(X(s),t-s,a_\star(s)) = \min_{a\in K(X(s),t-s)} c(s)b_{\mathcal{H}}(X(s),t-s,a).
$$

Therefore, using the two above informations, we have

$$
\int_0^t \mathbb{1}_{E^\eta_{\text{sing}}}(s)c(s)b^\varepsilon_{\mathcal{H}}(X^\varepsilon(s),t-s,a^\varepsilon(s))\mathbb{1}_{\{X^\varepsilon\in\mathcal{H}\}}(s)\,\mathrm{d}s \ge \int_0^t \mathbb{1}_{E^\eta_{\text{sing}}}(s)c(s)b_{\mathcal{H}}(X(s),t-s,a_\star(s))\mathbb{1}_{\{X^\varepsilon\in\mathcal{H}\}}(s)\,\mathrm{d}s + o_\varepsilon(1)
\tag{5.8}
$$

Now we can pass to the weak limit in (5.7)-(5.8) using the measures ν_i and $\nu_{\mathcal{H}}$. We obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0} I^\varepsilon \ge &\int_0^t c(s)\mathbb{1}_{E^\eta_{\text{sing}}}(s)\left(\sum_{i=1,2}\int_{A_i} b_i\big(X(s),t-s,\alpha_i(s)\big)\nu_i(s,\,\mathrm{d}\alpha_i) + \int_A b_{\mathcal{H}}(X(s),t-s,a_\star(s))\nu_{\mathcal{H}}(s,da)\right)\mathrm{d}s \\
&+\int_0^t d(s)\mathbb{1}_{E^\eta_{\text{sing}}}(s)\,\mathrm{d}s \\
=&\int_0^t \mathbb{1}_{E^\eta_{\text{sing}}}(s)\Psi_s\left(\sum_{i=1,2}\int_{A_i} b_i\big(X(s),t-s,\alpha_i(s)\big)\nu_i(s,\,\mathrm{d}\alpha_i) + \int_A b_{\mathcal{H}}(X(s),t-s,a_\star(s))\nu_{\mathcal{H}}(s,da)\right)\mathrm{d}s\,.
\end{aligned}
$$

Next we remark that, by (5.6), for $i=1,2$

$$
\int_{A_i} b_i\big(X(s),t-s,\alpha_i(s)\big)\nu_i(s,\,\mathrm{d}\alpha_i) = b_i\big(X(s),\sigma(s),\alpha_i^\sharp(s)\big)\pi_i(s)\in K(X(s),t-s)
$$

and $b_{\mathcal{H}}(X(s),t-s,a_\star(s))\in K(X(s),t-s)$ by construction. Therefore, since $\nu_1(s,A_1)+\nu_2(s,A_2)+\nu_{\mathcal{H}}(s,A)=1$ and $K(X(s),t-s)$ is convex, we have

$$
\Psi_s\left(\sum_{i=1,2}\int_{A_i} b_i\big(X(s),t-s,\alpha_i\big)\nu_i(s,\,\mathrm{d}\alpha_i) + \int_A b_{\mathcal{H}}(X(s),t-s,a_\star)\nu_{\mathcal{H}}(s,da_\star)\right)\ge 1
$$

We end up with $\lim_{\varepsilon\to 0} I^\varepsilon \geq |E^\eta_{\rm sing}| > 0$ which is a contradiction with the fact that $\lim I^\varepsilon = -|E^\eta_{\rm sing}| < 0$ by assumption. This proves that for any $\eta > 0$, $|E^\eta_{\rm sing}| = 0$ and we deduce that for almost any s, the limit dynamic $b_\mathcal{H}\big(X(s), t-s, a(s)\big)$ is regular, which ends the proof.

Proof of *iii)* — This result follows by remarking that the arguments above holds true also is we consider a sequence $(x_\varepsilon, t_\varepsilon) \to (x,t)$ as $\varepsilon \to 0$. We decided not to write it directly in the general case for the sake of simplicity. $\square$

Remark 5.4. *Through the above proof, it can be easily seen that this stability result extends to the case when the domain depends on ε : indeed the proof is done using $(\mathbf{H}_\Omega)$, reducing to the case when $\mathcal{H} = \{x_N = 0\}$ through Assumption $(\mathbf{H}^{x_0}_\Omega)$. To extend the result, we have to suppose that the $\Omega^\varepsilon_1, \Omega^\varepsilon_2$ converges in a C^1-sense to Ω_1, Ω_2 which means that the Ψ_ε in $(\mathbf{H}^{x_0}_\Omega)$ have to converge in C^1. Note that, this convergence has to be assumed $W^{2,\infty}$ if the required result is the convergence of solutions (instead of only sub or supersolution).*

Finally, we have a stability result for the maximal and minimal solutions:

Theorem 5.5. *Let us assume the hypotheses of Theorem 5.1. Then the associated value functions $\mathbf{U}^-_\varepsilon$ and $\mathbf{U}^+_\varepsilon$ converge respectively to $\mathbf{U}^-$ and $\mathbf{U}^+$.*

Proof. Let us first remark that the convergence of $\mathbf{U}^-_\varepsilon$ to $\mathbf{U}^-$ follows classically from the stability and comparison results Theorem 5.1 and Theorem 4.4. Moreover, the same results ensure us that $\mathbf{U}^+ \geq \limsup^* \mathbf{U}^+_\varepsilon$. Indeed, we only now that $\mathbf{U}^+$ is the maximal subsolution of (5.2), therefore the stability can be applied only to the subsolutions inequality.

In order to conclude we need to prove that $\mathbf{U}^+(x,t) \leq \liminf^* \mathbf{U}^+_\varepsilon(x,t)$ for all $(x,t) \in \mathbb{R}^N \times [0,T]$. For each $\varepsilon > 0$, there exists a $(X^\varepsilon, a^\varepsilon) \in \mathcal{T}^{\rm reg}_{x_\varepsilon, t_\varepsilon}$ such that

$$\mathbf{U}^+_\varepsilon(x_\varepsilon, t_\varepsilon) = J^\varepsilon(x_\varepsilon, t_\varepsilon; (X^\varepsilon, a^\varepsilon))$$

and we first consider a subsequence $(X^{\varepsilon_n}, a^{\varepsilon_n})$ such that $\liminf \mathbf{U}^+_\varepsilon(x_\varepsilon, t_\varepsilon) = \lim \mathbf{U}^+_{\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n})$. Then we use Lemma 5.3, parts *iii)*: up to another extraction, we may assume that $\mathbf{U}^+_{\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n}) = J^{\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n}; (X^{\varepsilon_n}, a^{\varepsilon_n})) \to J(x,t;(X,a))$ for some $(X,a) \in \mathcal{T}^{\rm reg}_{x,t}$. Hence,

$$\liminf \mathbf{U}^+_\varepsilon(x_\varepsilon, t_\varepsilon) = J(x,t;(X,a)) \geq \inf_{(X,a)\in\mathcal{T}^{\rm reg}_{x,t}} J(x,t;(X,a)) = \mathbf{U}^+(x,t)\,,$$

which ends the proof. $\square$

6 Further Remarks and Extensions

The simplified (but relevant) framework we describe above can be extended in several directions and we start by remarks concerning the different regions (Ω_1, Ω_2).

Because of the regularity assumptions we impose on the interfaces, there is no difference between $(\mathbf{H}_\Omega)$ and using a possibly infinite number regular open subsets $(\Omega_i)_i$ with either $1 \leq i \leq K$ or $i \in \mathbb{N}$ and satisfying the following assumptions

($\mathbf{H}_\Omega'$) *For all $i \neq j$, $\Omega_i \cap \Omega_j = \emptyset$ and $\mathbb{R}^N = \bigcup_i \overline{\Omega_i}$; for any $z \in \mathcal{H} := \mathbb{R}^N \setminus \left(\bigcup_i \Omega_i\right)$, there exist exactly two indices i,j such that $z \in \overline{\Omega_i} \cap \overline{\Omega_j} := \Gamma_{\{i,j\}}$. Moreover $\mathcal{H} := \bigcup_{i,j} \Gamma_{\{i,j\}}$ is C^1 in the controllable case and $W^{2,\infty}$ in the non-controllable case, (i.e. when there is only controllability in the normal direction).*

Concerning the regularity assumption on $\mathcal{H}$, we point out that, since our key arguments are local, we are always in a two-domains framework and even in a two-mains framework with a flat interface. This is why we have chosen to present the paper with just two domains Ω_1 and Ω_2. On the other hand, this regularity is used through some change of variable and it is necessary in order that the transformed Hamiltonians satisfy the right assumptions to prove the comparison result. In the controllable case, the solutions are Lipschitz continuous and it could be enough to have continuous b_i's and a C^1 change preserves this property. On the contrary, in the non-controllable case, the solutions may be just semi-continuous and the Lipschitz continuity of the b_i's is necessary. Here we need a $W^{2,\infty}$ change to preserve this property.

Because of the same argument, the Ω_i may depend on t and (this is an other way to formulate it) even we may assume that the Ω_i are domains in $\mathbb{R}^N \times (0,T)$ with the same regularity assumption as the one we use above (one has just to use ($\mathbf{H}_\Omega'$) with $\mathbb{R}^N$ being replaced by $\mathbb{R}^N \times (0,T)$). This is a consequence of the fact that, through our change of variable, t and the tangential coordinates on $\mathcal{H}$ play the same role. A corollary of this remark is that if $\mathbf{n}_i(\cdot) = (n_i^x, n_i^t) \in \mathbb{R}^N \times \mathbb{R}$ is the unit normal vector pointing outwards defined on $\partial\Omega_i$, then we have to assume $n_i^x \neq 0$. This is required to avoid, for example, the pathological situation of $\Omega_i \subset\subset \mathbb{R}^N \times (0,T)$.

As far as the control problem is concerned, it is clear from the proof that we can take into account without any difficulty : (i) general discount factors $(c_i(x,t,\alpha_i))$, (ii) infinite horizon control problem with multiple domains in the non-controllable case (extending the results of [6]) and (iii) the case where one has an additional control problem on $\mathcal{H}$: here it suffices to check that the proof of Theorem 3.8 (of [6, Thm. 3.3]) extends to this case. To do so, we make two remarks

(a) The control problem on $\mathcal{H}$ is associated to an Hamiltonian G and (3.15) should be replaced by

$$\max(\phi_t(x,t) + H_T\big(x,t,D_\mathcal{H}\phi(x,t)\big), \phi_t(x,t) + G\big(x,t,D_\mathcal{H}\phi(x,t)\big)) \geq 0 .$$

(b) The proof is going to consider (in the flat boundary case)

$$\begin{aligned}\varphi(\delta) :=& \max\{\phi_t(x,t) + H_1(x_0, v(x_0), D_\mathcal{H}\phi(x_0') + \delta e_N), \phi_t(x,t) + H_2(x_0, v(x_0), D_\mathcal{H}\phi(x_0') + \delta e_N), \\ & \phi_t(x,t) + G\big(x,t,D_\mathcal{H}\phi(x,t) + \delta e_N\big)\}\end{aligned}$$

but $\phi_t(x,t) + G\big(x,t,D_\mathcal{H}\phi(x,t) + \delta e_N\big) = \phi_t(x,t) + G\big(x,t,D_\mathcal{H}\phi(x,t)\big)$ since the G-Hamiltonian takes only into account the tangential part of the gradient and this quantity can be assumed to be strictly negative, otherwise we would be done. Therefore we see that the G-term plays no role in the proof.

To conclude, let us mention that the (interesting) cases of non-smooth $\mathcal{H}$ where the different regions can be separated by triple junction or the case of chessboard situations are still (far) out of the scope of this article.

7 Appendix: the flat interface case

In this appendix, we assume that we are in a local "flat" situation. More precisely, we denote by $\tilde{\Omega}$ a bounded open subset of $\mathbb{R}^N$ (we actually have in mind the image of a ball $B(x,r)$ by a diffeomorphism ψ which purpose is to flatten the interface). We assume that $0 \in \tilde{\Omega}$ and consider

$$\tilde{\Omega}_1 = \{x_N > 0\} \cap \tilde{\Omega}\,,\ \tilde{\Omega}_2 = \{x_N < 0\} \cap \tilde{\Omega}\,.$$

We use the notations $\tilde{\Gamma} := \partial\tilde{\Omega}_1 \cap \partial\tilde{\Omega}_2 = \tilde{\Omega} \cap \{x_N = 0\}$, so that $\tilde{\Omega} = \tilde{\Omega}_1 \cup \tilde{\Omega}_2 \cup \tilde{\Gamma}$. Following Section 4, for $0 < h < t_0 < T$, we denote by $\tilde{Q} := \tilde{\Omega} \times (t_0 - h, t_0)$ and $\partial_p \tilde{Q} = \tilde{\Omega} \times \{t_0 - h\} \cup \partial\tilde{\Omega} \times [t_0 - h, t_0]$ its parabolic boundary. We also denote by $\mathbf{e}_N$ the N-th unit vector in $\mathbb{R}^N$.

For $i = 1, 2$, we are given dynamics $\tilde{b}_i$ and costs $\tilde{l}_i$ in each $\tilde{\Omega}_i$ and we define $\tilde{H}_i$, $\tilde{H}_T$, $\tilde{H_T^{\mathrm{reg}}}$ exactly as we did for the same Hamiltonians without the tilde. With the convention of Section 3, this allows us to consider the problem

$$\tilde{w}_t + \tilde{\mathbb{H}}^-(x,t,Dw) = 0 \quad \text{in } \tilde{Q}\,. \tag{7.1}$$

In all the following we assume that the dynamics and costs $\tilde{b}_i, \tilde{l}_i$ satisfy $(\mathbf{H}_{\mathrm{C}})$ with constants denoted with a tilde: $\tilde{M}_b$, $\tilde{L}_b$, $\tilde{M}_l$, $\tilde{m}_l$ and $\tilde{\delta}$. Of course, this is the case after our reduction to the flat case if the b_i and l_i satisfy $(\mathbf{H}_{\mathrm{C}})$. Before proving the local comparison result which is the main result of this appendix, we need first to obtain some properties of the Hamiltonians.

Appendix A. Properties of the Hamiltonians

To begin with, we prove that the normal controllability assumption $(\mathbf{H}_{\mathrm{C}}^4)$ gives coercivity in the p_N-variable:

Lemma 7.1. *Assume that the dynamics $\tilde{b}_i$ and costs $\tilde{l}_i$ satisfy $(\mathbf{H}_{\mathrm{C}})$. Then, there exists a constant $\tilde{C}_M$ such that, for $i = 1..2$ and $p = (p', p_N)$, we have*

$$\tilde{H}_i(x,t,p) \geq \tilde{\delta}|p_N| - \tilde{C}_M(1 + |p'|)\ ,$$

where δ is given by assumption $(\mathbf{H}_{\mathrm{C}}^4)$ and $\tilde{C}_M = \max\{\tilde{M}_b, \tilde{M}_l\}$ in $(\mathbf{H}_{\mathrm{C}}^1)$ and $(\mathbf{H}_{\mathrm{C}}^2)$.

Proof. We provide the proof in the case of $\tilde{H}_1$, it is similar for $\tilde{H}_2$. The (partial) controlability assumption $(\mathbf{H}_{\mathrm{C}}^4)$ implies the existence of controls $\alpha_1, \alpha_2 \in A_1$ such that

$$-\tilde{b}_1(x,t,\alpha_1) \cdot \mathbf{e}_N = \tilde{\delta} > 0\ ,\ -\tilde{b}_1(x,t,\alpha_2) \cdot \mathbf{e}_N = -\tilde{\delta}\,.$$

Now we compute $\tilde{H}_1(x,t,p)$ assuming that $p_N > 0$ (the other case is treated similarly).

$$\begin{aligned}
\tilde{H}_1(x,t,p) &\geq -\tilde{b}_1(x,t,\alpha_1) \cdot p - \tilde{l}_1(x,t,\alpha_1) \\
&\geq -\tilde{b}_1(x,t,\alpha_1) \cdot (p' + p_N \mathbf{e}_N) - \tilde{l}_1(x,t,\alpha_1) \\
&\geq \tilde{\delta} p_N - \tilde{b}_1(x,t,\alpha_1) \cdot p' - \tilde{l}_1(x,t,\alpha_1) \\
&\geq \tilde{\delta} p_N - \tilde{C}_M |p'| - \tilde{C}_M\ ,
\end{aligned}$$

the last line coming from the boundedness of $\tilde{b}_1$ and $\tilde{l}_1$. This concludes the proof. $\square$

Let us now give the needed regularity properties of the tangential Hamiltonian H_T. We do the proof in the non-flat case for the sake of completeness.

Lemma 7.2. *Assume* $(\mathbf{H}_\Omega)$ *and* $(\mathbf{H}_\mathrm{C})$. *The tangential Hamiltonian defined in* (3.1) *satisfies the following Lipschitz property: Moreover, for any* $(z, p_\mathcal{H}), (z', q_\mathcal{H}) \in T\mathcal{H}$ *and* $t, t' \in [0, T]$

$$|H_T(z,t,p_\mathcal{H}) - H_T(z',t',q_\mathcal{H})| \le M|(z,t)-(z',t')|\big(|p_\mathcal{H}|+|q_\mathcal{H}|\big) + M_b|p_\mathcal{H}-q_\mathcal{H}| + m(|(z,t)-(z',t')|)\,, \tag{7.2}$$

where, if $M_b, M_l, L_b, m_l, \delta$ *are given by* $(\mathbf{H}^1_\mathrm{C})$ *and* $(\mathbf{H}^2_\mathrm{C})$,

$$M := (L_b + 2M_b(L_b + M_b L_\mathbf{n})\delta^{-1})\,,$$

$L_\mathbf{n}$ *being the Lipschitz constant of* $\mathbf{n}_1$ *and*

$$m(t) = (L_b + 2M_l\bar{C}\delta^{-1})t + m_l(t) \quad \text{for } t \ge 0\,.$$

Proof. We only recal that $p_\mathcal{H}$ can be considered at the same time as a vector in $T_z\mathcal{H}$ (of dimension $(N-1)$) and a vector in $\mathbb{R}^N$ by using $(p_\mathcal{H}, 0)$ where the zero means "$0\mathbf{n}_1(z)$". Then $\big\langle P_z b_\mathcal{H}(z,t,a), p_\mathcal{H}\big\rangle = b_\mathcal{H}(z,t,a)\cdot p_\mathcal{H}$ with a slight abuse of notations. With this in mind, the proof easily follows from Lemma 7.4 below and standard arguments. □

Remark 7.3. *In various proofs, we extend a test function from* $\mathcal{H}$ *to* $\mathbb{R}^N$, *which gives a* N*-dimensional vector* $p = D\phi$. *Then, to test* H_T *we have to compute the tangential projections on* $\mathcal{H}$: $p_\mathcal{H} = P_z p$ *and* $q_\mathcal{H} = P_{z'}p$ *which of course may not be the same, reflecting the possibly non-flat geometry of* $\mathcal{H}$. *Hence the term* $M_b|p_\mathcal{H} - q_\mathcal{H}|$ *has to be dealt with even if we start from the same vector* $p \in \mathbb{R}^N$ *for both points* z, z'.

Lemma 7.4. *Assume* $(\mathbf{H}_\Omega)$ *and* $(\mathbf{H}_\mathrm{C})$. *For any* $(z,t), (z',t') \in \mathcal{H}\times[0,T]$ *and for each control* $a \in A_0(z,t)$, *there exists a control* $a' \in A_0(z',t')$ *such that, if* $\bar{C} := L_b + M_b L_\mathbf{n}$

$$|b_\mathcal{H}(z,t,a) - b_\mathcal{H}(z',t',a'))| \le (L_b + 2M_b\bar{C}\delta^{-1})|(z,t)-(z,t')|$$
$$|l_\mathcal{H}(z,t,a) - l_\mathcal{H}(z',t',a'))| \le 2M_l\bar{C}\delta^{-1}|(z,t)-(z,t')| + m_l(|(z,t)-(z,t')|)\,.$$

Proof. Let us consider a control $a \in A_0(z,t)$, i.e. $b_\mathcal{H}(z,t,a)\cdot\mathbf{n}_1(z) = 0$. Fix $(z',t') \in \mathcal{H}\times[0,T]$, we have two possibilities. If $b_\mathcal{H}(z',t',a)\cdot\mathbf{n}_1(z') = 0$ the conclusion easily follows because $a' = a \in \mathcal{A}_0(z',t')$ and

$$|b_\mathcal{H}(z,t,a) - b_\mathcal{H}(z',t',a)| \le L_b|(z,t)-(z',t')|\,, \tag{7.3}$$
$$|l_\mathcal{H}(z,t,a) - l_\mathcal{H}(z',t',a)| \le m_l(|(z,t)-(z',t')|)\,. \tag{7.4}$$

Otherwise $b_\mathcal{H}(z',t',a)\cdot\mathbf{n}_1(z') \ne 0$. Let us suppose, for example, that $b_\mathcal{H}(z',t',a)\cdot\mathbf{n}_1(z) > 0$ (for the other sign the same argument will apply so we will not detail it). We first remark that by $(\mathbf{H}^1_\mathrm{C})$

$$|b_\mathcal{H}(z',t',a)\cdot\mathbf{n}_1(z')| = |b_\mathcal{H}(z',t',a)\cdot\mathbf{n}_1(z') - b_\mathcal{H}(z,t,a)\cdot\mathbf{n}_1(z)| \le \bar{C}|(z,t)-(z',t')| \tag{7.5}$$

with $\bar{C} := L_b + M_b L_\mathbf{n}$. By the controllability assumption in $(\mathbf{H}^4_\mathrm{C})$ there exists a control $a_1 \in A$ such that $b_\mathcal{H}(z',t',a_1)\cdot\mathbf{n}_1(z') = -\delta\mathbf{n}_1(z')$. We then set

$$\bar{\mu} := \frac{\delta}{b_\mathcal{H}(z',t',a)\cdot\mathbf{n}_1(z') + \delta},$$

since $\bar{\mu} \in]0,1[$, by the convexity assumption in $(\mathbf{H}_{\mathrm{C}}^3)$, the exists a control a' such that

$$\bar{\mu}(b_{\mathcal{H}}(z',t',a), l_{\mathcal{H}}(z',t',a)) + (1-\bar{\mu})(b_{\mathcal{H}}(z',t',a_1), l_{\mathcal{H}}(z',t',a_1)) = (b_{\mathcal{H}}(z',t',a'), l_{\mathcal{H}}(z',t',a')).$$

By construction $b_{\mathcal{H}}(z',t',a') \cdot \mathbf{n}_1(z') = 0$, therefore $a' \in A_0(z',t')$. Moreover, since

$$(1-\bar{\mu}) = \frac{b_{\mathcal{H}}(z',t',a) \cdot \mathbf{n}_1(z')}{b_{\mathcal{H}}(z',t',a) \cdot \mathbf{n}_1(z') + \delta}$$

by (7.5), we have

$$|b_{\mathcal{H}}(z',t',a) - b_{\mathcal{H}}(z',t',a')| \leq (1-\bar{\mu})|b_{\mathcal{H}}(z',t',a) - b_{\mathcal{H}}(z',t',a_1)| \leq 2M_b\bar{C}\delta^{-1}|(z,t)-(z',t')|\ ,$$

and the same inequality holds for $l_{\mathcal{H}}$, replacing M_b by M_l. Hence, thanks to (7.3)-(7.4), we obtain

$$\begin{aligned} |b_{\mathcal{H}}(z,t,a) - b_{\mathcal{H}}(z',t',a'))| &\leq (L_b + 2M_b\bar{C}\delta^{-1})|(z,t)-(z',t')| \\ |l_{\mathcal{H}}(z,t,a) - l_{\mathcal{H}}(z',t',a'))| &\leq 2M_l\bar{C}\delta^{-1}|(z,t)-(z',t')| + m_l(|(z,t)-(z',t')|)\ , \end{aligned}$$

and this concludes the proof. ☐

Remark 7.5. *The results of Lemma 7.2 and 7.4 still hold in the case of H_T^{reg}, changing the constants in* (7.2) *and in the result of Lemma 7.4. The simplest way to prove it is the following : we only do it for b_1, b_2 but a correct argument would require a proof in $(b_1,l_1),(b_2,l_2)$. We first remark that if*

$$b_{\mathcal{H}}(z,t,a) = \mu b_1(z,t,\alpha_1) + (1-\mu)b_2(z,t,\alpha_2)\,,$$

and if $|(z,t)-(z',t')|$ is small enough, we may assume without loss of generality that, for $i=1,2$,

$$b_i(z,t,\alpha_1)\cdot \mathbf{n}_i(z) \geq 3(L_b + 2M_b\bar{C}\delta^{-1})|(z,t)-(z',t')|\ . \tag{7.6}$$

Indeed, by the controllability assumption in $(\mathbf{H}_{\mathrm{C}}^4)$, there exists a control $\hat{\alpha}_i \in A_i$ such that $b_i(z,t,\hat{\alpha}_i)\cdot \mathbf{n}_i(z) = \delta \mathbf{n}_i(z)$. Then, by taking $|(z,t)-(z',t')|$ small enough, we can always assume that $3(L_b + 2M_b\bar{C}\delta^{-1})|(z,t)-(z',t')|$ is between $b_i(z,t,\hat{\alpha}_i)\cdot\mathbf{n}_i(z)$ and $b_i(z,t,\alpha_i)\cdot\mathbf{n}_i(z)$. We can then choose $\mu_i \in [0,1]$ such that

$$(\mu_i b_i(z,t,\alpha_i) + (1-\mu_i)b_i(z,t,\hat{\alpha}_i))\cdot \mathbf{n}_i(z) = 3(L_b + 2M_b\bar{C}\delta^{-1})|(z,t)-(z',t')|\ .$$

Finally Assumption $(\mathbf{H}_{\mathrm{C}}^3)$ ensures that there exists controls $\tilde{\alpha}_i$ such that

$$b_i(z,t,\tilde{\alpha}_i) = \mu_i b_i(z,t,\alpha_i) + (1-\mu_i)b_i(z,t,\hat{\alpha}_i)\ .$$

To obtain a new $b_{\mathcal{H}}(z,t,\tilde{a})$, we choose $\tilde{\mu} \in [0,1]$ such that

$$[\tilde{\mu} b_1(z,t,\tilde{\alpha}_1) + (1-\tilde{\mu})b_2(z,t,\tilde{\alpha}_2)]\cdot \mathbf{n}_1(z) = 0\,.$$

To conclude we remark that a careful examination of the estimate on $\bar{\mu}$ in the proof of Lemma 7.4 shows that, if we start from a control $\tilde{a} \in A_0^{\mathrm{reg}}(z,t)$ verifying (7.6) *the associated control $\tilde{a}' \in A_0(z',t')$ is in fact in $A_0^{\mathrm{reg}}(z',t')$.*

Remark 7.6. *If the b_i are only assumed to be continuous, we have similar estimates involving the modulus of contuity m_b instead of the Lipschitz constant L_b (as we did for the l_i with m_l).*

Appendix B. the local comparison result

Lemma 7.7. *Assume that the dynamics $\tilde{b}_i$ and costs $\tilde{l}_i$ satisfy* $(\mathbf{H}_{\mathrm{C}})$. *If $\tilde{u}$ is an usc subsolution of (7.1) and $\tilde{v}$ a lsc supersolution of (7.1), then*

$$\|(\tilde{u}-\tilde{v})_+\|_{L^\infty(\tilde{Q})} \le \|(\tilde{u}-\tilde{v})_+\|_{L^\infty(\partial_p\tilde{Q})} \,. \tag{7.7}$$

Proof. As in [6] the first steps consist in regularizing the subsolution. To do so, depending on the context, we write either x or (x', x_N) where $x' \in \mathbb{R}^{N-1}$ for a point in $\tilde{\Omega}$. Moreover, for the sake of simplicity, we will use both notations: $H(x,t,p)$ or $H(x', x_N, t, p)$.

Step 1 — We first define the sup-convolution in time and in the x'-variable for $\tilde{u}$ as follows

$$\tilde{u}_\alpha(x,t) := \max_{y',t'} \left\{ \tilde{u}(y', x_N, t') - \exp(Kt)\left(\frac{|x'-y'|^2}{\alpha^2} + \frac{|t-t'|^2}{\alpha^2}\right)\right\}$$

where the maximum is taken over all y', t' such that $(y', x_N, t') \in \overline{\tilde{Q}}$ and where K is a large positive constant to be chosen later. By the definition of the supremum, if it is achived at y', t', we have

$$\tilde{u}_\alpha(x,t) = \tilde{u}(y', x_N, t') - \exp(Kt)\left(\frac{|x'-y'|^2}{\alpha^2} + \frac{|t-t'|^2}{\alpha^2}\right) \le \tilde{u}(x,t) \,,$$

and therefore (since $K > 0$), $\frac{|x'-y'|^2}{\alpha^2} + \frac{|t-t'|^2}{\alpha^2} \le 2||u||_\infty$. Since we want to use viscosity inequalities for u at (y', x_N, t'), we need these points to be in $\tilde{Q}$ and thanks to the above inequality, in order to do it, we have to restrict (x,t) to be in

$$\tilde{Q}_\alpha := \left\{ x \in \tilde{\Omega} :\ \mathrm{dist}(x, \partial\tilde{\Omega}) > (2||\tilde{u}||_\infty)^{1/2}\alpha \right\} \times \left(t_0 - h + (2||\tilde{u}||_\infty)^{1/2}\alpha, t_0 - (2||\tilde{u}||_\infty)^{1/2}\alpha \right) \,.$$

Our result on $\tilde{u}_\alpha$ is the

Lemma 7.8. *The Lipschitz continuous function $\tilde{u}_\alpha$ satisfies $(\tilde{u}_\alpha)_t + \tilde{\mathbb{H}}^-(x,t,D\tilde{u}_\alpha) \le m(\alpha)$ in $\tilde{Q}_\alpha$ for some $m(\alpha)$ converging to 0 as α tends to 0.*

Proof. We first remark that $\tilde{u}_\alpha$ is Lipschitz continuous with respect to time t and to the x'-variable by the classical properties of the sup-convolution. Once we know that $(\tilde{u}_\alpha)_t$ and $D_{x'}\tilde{u}_\alpha$ are bounded, the Lipschitz continuity with respect to the x_N-variable comes from the fact that $\tilde{u}_\alpha$ is a subsolution of the $\tilde{\mathbb{H}}^-$-equation thanks to the coerciveness of the Hamiltonian in the p_n-variable given by Lemma 7.1. Indeed, by applying formally Lemma 7.1

$$(\tilde{u}_\alpha)_t + \tilde{\delta}\big|\partial_{x_N}\tilde{u}_\alpha\big| - \tilde{C}_M(1 + |D_{x'}\tilde{u}_\alpha|) \le m(\alpha) \quad \text{in } \tilde{Q}_\alpha \,,$$

a claim which can be justified by very classical arguments.

To check that it is a subsolution of the $\tilde{\mathbb{H}}^-$-equation, we consider a test-function ϕ and a point (x,t) where $\tilde{u}_\alpha - \phi$ reaches a local maximum. Then considering a maximum in (z,s) of $\tilde{u}_\alpha(z,s) - \phi(z,s)$ leads us to consider a maximum in (z,s,y',t') of $\tilde{u}(y', z_N, t') - \exp(Ks)\left(\frac{|z'-y'|^2}{\alpha^2} + \frac{|s-t'|^2}{\alpha^2}\right) - \phi(z,s)$. If

$$\tilde{u}_\alpha(x,t) := \tilde{u}(y', x_N, t') - \exp(Kt)\left(\frac{|x'-y'|^2}{\alpha^2} + \frac{|t-t'|^2}{\alpha^2}\right) \,,$$

(we still write y', t' for the variables where the max is attained for simplicity of notations) we deduce several things : first, we have a max in z' and s which gives

$$
\begin{aligned}
D_{x'}\phi(x', x_N, t) &= \frac{2(y'-x')}{\alpha^2}\exp(Kt)\,,\\
\phi_t(x', x_N, t) &= \frac{2(t'-t)}{\alpha^2}\exp(Kt) - K\exp(Kt)\left(\frac{|x'-y'|^2}{\alpha^2} + \frac{|t-t'|^2}{\alpha^2}\right).
\end{aligned}
$$

Then, if $x_N > 0$, we write down the viscosity inequality for $\tilde{u}$ and $\tilde{H}_1$, the proof being similar for $\tilde{H}_2$ if $x_N < 0$ and $\tilde{H}_T$ if $x_N = 0$ thanks to Lemma 7.2 below.
Using as test function $(y', x_N, t') \mapsto \phi(x', x_N, t') + \exp(Kt)\left(\frac{|x'-y'|^2}{\alpha^2} + \frac{|t-t'|^2}{\alpha^2}\right)$, we have

$$
\frac{2(t'-t)}{\alpha^2}\exp(Kt) + \tilde{H}_1\left(y', x_N, t', \frac{2(y'-x')}{\alpha^2}\exp(Kt) + \partial_{x_N}\phi(x', x_N, t)\,\mathbf{e}_N\right) \leq 0\,. \tag{7.8}
$$

Notice that, combining the previous results, we have

$$
\phi_t(x,t) + K\exp(Kt)\left(\frac{|x'-y'|^2}{\alpha^2} + \frac{|t-t'|^2}{\alpha^2}\right) + \tilde{H}_1\big(y', x_N, t', D\phi(x,t)\big) \leq 0\,.
$$

In order to obtain the right inequality, we have to change y' in x' and t' in t. The only difficulty for doing it, compared to the usual arguments, is the $\partial_{x_N}\phi(x', x_N, t)$-term in (7.8) which we need to control. Using the lemma for (7.8) yields

$$
\left|\partial_{x_N}\phi\right| \leq \tilde{\delta}^{-1}\left(\tilde{C}_M\Big(\frac{2|y'-x'|}{\alpha^2}\exp(Kt) + 1\Big) + \frac{2|t'-t|}{\alpha^2}\exp(Kt)\right)\,. \tag{7.9}
$$

On the other hand, by the Lipschitz continuity of $\tilde{b}_1$ and the continuity of $\tilde{l}_1$, (in $(\mathbf{H}_{\mathrm{C}}^2)$) we have

$$
\big|\tilde{H}_1(y', x_N, t', p) - \tilde{H}_1(x, t, p)\big| \leq \tilde{L}_b(|y'-x'| + |t'-t|)|p| + \tilde{m}_l(|y'-x'| + |t'-t|)\,.
$$

Hence $\phi_t(x,t) + \tilde{H}_1\big(x, t, D\phi\big) \leq r.h.s$, where

$$
\begin{aligned}
r.h.s := {}& -K\exp(Kt)\left(\frac{|x'-y'|^2}{\alpha^2} + \frac{|t-t'|^2}{\alpha^2}\right) + \tilde{L}_b(|y'-x'| + |t'-t|)\Big(\frac{2|y'-x'|}{\alpha^2}\exp(Kt) + \left|\partial_{x_N}\phi\right|\Big)\\
&+ \tilde{m}_l(|y'-x'| + |t'-t|)\,.
\end{aligned}
$$

Therefore, thanks to (7.9),

$$
\begin{aligned}
r.h.s \;\leq\; & -K\exp(Kt)\left(\frac{|x'-y'|^2}{\alpha^2} + \frac{|t-t'|^2}{\alpha^2}\right) + \tilde{L}_b\exp(Kt)\big(|y'-x'| + |t'-t|\big)\frac{2|y'-x'|}{\alpha^2}\\
&+ \frac{\tilde{L}_b\exp(Kt)}{\tilde{\delta}}\big(|y'-x'| + |t'-t|\big)\left(\tilde{C}_M\frac{2|y'-x'|}{\alpha^2} + \frac{2|t'-t|}{\alpha^2}\right)\\
&+ \frac{\tilde{L}_b\tilde{C}_M}{\tilde{\delta}}\big(|y'-x'| + |t'-t|\big) + \tilde{m}_b\big(|y'-x'| + |t'-t|\big)\,.
\end{aligned}
$$

Since by construction $|y'-x'| + |t'-t| \leq 2(2||\tilde{u}||_\infty)^{1/2}\alpha$ the last line gives the $m(\alpha)$ which appears in the statement of Lemma 7.8. For the other terms, tedious but straightforward computations and the use of Cauchy-Schwarz inequality show that they give a negative contribution provided K is big enough. And the proof of Lemma 7.8 is complete. $\square$

STEP 2 — Then, for $\varepsilon \ll 1$, we introduce the function $\tilde u_\alpha^\varepsilon := \tilde u_\alpha * \rho_\varepsilon - [m(\alpha)+\tilde m(\varepsilon)]t$ where $m(\alpha)$ appears in the statement of Lemma 7.8, $\tilde m(\varepsilon)$ is a quantity to be chosen later which converges to 0 when $\varepsilon \to 0$ and $\rho_\varepsilon(x',t)$ is a standard (positive) mollifying kernel defined on $\mathbb{R}^{N-1}\times[0,T]$ as follows

$$\rho_\varepsilon(x',t) = \frac{1}{\varepsilon^{N-1}}\rho(\frac{x'}{\varepsilon},\frac{t}{\varepsilon})\ ,$$

where $\rho \in C^\infty(\mathbb{R}^{N-1}\times[0,T])$, $\displaystyle\int_{\mathbb{R}^{N-1}\times[0,T]}\rho(y)dy = 1$, and $\operatorname{supp}\{\rho\} = B_{\mathbb{R}^{N-1}\times[0,T]}(0,1)$.

We assume that the support of ρ_ε is the ball $B(0,\varepsilon)$ so that again, we define the convolution only in

$$\tilde Q_{\alpha,\varepsilon} := \left\{x\in\tilde\Omega:\ \operatorname{dist}(x,\partial\tilde\Omega) > (2||\tilde u||_\infty)^{1/2}\alpha+\varepsilon\right\}\times\Big(t_0-h+(2||\tilde u||_\infty)^{1/2}\alpha+\varepsilon, t_0-(2||\tilde u||_\infty)^{1/2}\alpha\Big)\ .$$

Lemma 7.9. *For any $\varepsilon \ll 1$, there exists $\tilde m(\varepsilon)$ such that $\tilde m(\varepsilon)\to 0$ as $\varepsilon\to 0$ and the function $\tilde u_\alpha^\varepsilon$ satisfies $(\tilde u_\alpha^\varepsilon)_t + \tilde{\mathbb{H}}^-(x,t,D\tilde u_\alpha^\varepsilon)\le 0$ in $\tilde Q_{\alpha,\varepsilon}$.*

We skip the proof of this lemma which is analogous to the corresponding one in [6, Lemma 4.2] since $\tilde u_\alpha$ is Lipschitz continuous. We just point out that $\tilde m(\varepsilon)$ comes from (and is used to control) the error in the convolution procedure.

STEP 3 — We are now able to prove the comparison result for $\tilde u$ and $\tilde v$ in $\tilde Q$. For a fixed pair (α,ε), we have to argue in $\tilde Q_{\alpha,\varepsilon}$. First, we point out that for any $\eta>0$, $\tilde u_\alpha^\varepsilon - \eta t$ is C^1 with respect to time t and the $x_1,\dots,x_{N-1}$ variables and therefore on $\tilde\Gamma\cap\tilde Q_{\alpha,\varepsilon}$ it is both a test-function for the $\tilde v$-inequality and it satisfies a strict subsolution inequality in the classical sense. Thanks to Theorem 3.8 we can argue as in [6, Theorem 4.1] and conclude that $\tilde v-(\tilde u_\alpha^\varepsilon-\eta t)$ cannot achieve a minimum point in $\tilde\Gamma\cap\tilde Q_{\alpha,\varepsilon}$. Moreover, since $\tilde u_\alpha^\varepsilon-\eta t$ is a *strict* subsolution, in $\tilde\Omega_1\cap\tilde Q_{\alpha,\varepsilon}$ and $\tilde\Omega_2\cap\tilde Q_{\alpha,\varepsilon}$ the conclusion follows by standard arguments since we are dealing with a standard Hamilton-Jacobi Equation. Thus $\tilde v-(\tilde u_\alpha^\varepsilon-\eta t)$ cannot have a minimum point in $\tilde Q_{\alpha,\varepsilon}$ and this immediately yields

$$\|(\tilde u_\alpha^\varepsilon-\eta t-\tilde v)_+\|_{L^\infty(\tilde Q_{\alpha,\varepsilon})}\le\|(\tilde u_\alpha^\varepsilon-\eta t-\tilde v)_+\|_{L^\infty(\partial_p\tilde Q_{\alpha,\varepsilon})}\,.$$

Letting η tend to 0 we obtain $\|(\tilde u_\alpha^\varepsilon-\tilde v)_+\|_{L^\infty(\tilde Q_{\alpha,\varepsilon})}\le\|(\tilde u_\alpha^\varepsilon-\tilde v)_+\|_{L^\infty(\partial_p\tilde Q_{\alpha,\varepsilon})}$. In order to prove the final result, we have to pass to the limit as $\varepsilon\to 0$ and then as $\alpha\to 0$.

Letting ε tend to 0 is easy since $\tilde u_\alpha$ is continuous (we may even argue in a slightly smaller domain/cylinder). Therefore

$$\|(\tilde u_\alpha-m(\alpha)t-\tilde v)_+\|_{L^\infty(\tilde Q_\alpha)}\le\|(\tilde u_\alpha-m(\alpha)t-\tilde v)_+\|_{L^\infty(\partial_p\tilde Q_\alpha)}\,.$$

Fix now $\alpha_0>0$ and $(y,s)\in\tilde Q_{\alpha_0}$. For all $0<\alpha\le\alpha_0$ we have

$$(\tilde u_\alpha(y,s)-m(\alpha)t-\tilde v(y,s))_+\le\|(\tilde u_\alpha-m(\alpha)t-\tilde v)_+\|_{L^\infty(\partial_p\tilde Q_\alpha)}\,. \tag{7.10}$$

Let us observe that by the properties of the sup-convolution and the fact that $\tilde u$ is upper-semi-continuous we have that $\limsup_{\alpha\to 0}\|(\tilde u_\alpha-m(\alpha)t-\tilde v)_+\|_{L^\infty(\partial_p\tilde Q_\alpha)}\le\|(\tilde u-\tilde v)_+\|_{L^\infty(\partial_p\tilde Q)}$. Therefore, by the pointwise convergence of $\tilde u_\alpha\to\tilde u$, passing to the limsup in (7.10) we deduce

$$(\tilde u(y,s)-\tilde v(y,s))_+\le\|(\tilde u-\tilde v)_+\|_{L^\infty(\partial_p\tilde Q)}\quad\forall (y,s)\in\tilde Q_{\alpha_0}.$$

Since α_0 is arbitrary we get $\|(\tilde u-\tilde v)_+\|_{L^\infty(\tilde Q)}\le\|(\tilde u-\tilde v)_+\|_{L^\infty(\partial_p\tilde Q)}$ and the result is proved. □

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